A METHODOLOGY FOR INVESTIGATING TEACHERS' MATHEMATICAL MEANINGS FOR TEACHING[1]

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We first argue that a focus on teachers' mathematical meanings rather than on their mathematical knowledge holds greater potential for connecting research on what teachers know, what they do, and what students learn. We then describe a method to design items and instruments that assess teachers' mathematical meanings.

A number of recent studies attempt to assess mathematics teachers' *mathematical knowledge for teaching*. Many studies focused on knowledge for teaching K-8 mathematics (e.g., Ball, Thames, & Phelps, 2008; Bell, Wilson, Higgins, & McCoach, 2010; Carpenter, Fennema, Peterson, & Carey, 1988; Hill, Schilling, & Ball, 2004; Izsák, 2008). Other studies focused on knowledge for teaching secondary mathematics (e.g., Doerr, 2004; McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012; Steele, Hillen, & Smith, 2013). A commonality among studies of mathematical knowledge for teaching is that even though they aim to investigate teachers' mathematical knowledge, none of them says what they mean by knowledge, let alone draw from a theory of knowledge.

We view this as a worrisome state of affairs. Without a definition of knowledge, the main construct being investigated is open to the meaning of knowledge that a reader brings to reports of studies investigating it. Without a theory of knowledge and how it is attained, assessments of it have no way to meaningfully connect what teachers know (as determined by the assessment), the instructional decisions they make, and what students learn.

Without a theory of knowledge, assessments of it tend to assess teachers' performance instead of their thinking. For example, Steele, Hillen, and Smith (2013) identified knowledge of functions that secondary mathematics teachers should hold (Figure 1). The entries in their table describe things a teacher should be able to do, not what they should know. Granted, teachers must know something to be able to do these things, but what they might know is unspecified and how that knowledge might express itself in these behaviours is unexplained.

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Common content knowledge	Specialized content knowledge
State a definition of function	Evaluate function definitions and consider their utility for teaching
Create and classify examples and non-examples of function	Be able to create and move flexibly between representations of functions

Figure 1. Core aspects of knowledge of function (Steele, et al., 2013, p. 456)

Most importantly, without a theory of knowledge that addresses the relationship of teachers' knowledge to teachers' actions, and without a theory of how teachers' actions can impact student thinking, there is little basis for designing knowledge-oriented professional development that has the potential to actually help teachers help students learn what they hope students will know. Thompson (2013) argued and illustrated the claim that teachers convey their personal meanings to students, whether teachers hold them tacitly or consciously, and it is meanings students develop that are the foundation for their future uses of the mathematics they learn and are foundational for their future mathematical learning.

Our approach to assessing teachers' mathematical knowledge for teaching is first to eschew speaking of "knowledge" and instead focus on teachers' mathematical meanings. Thus, we named our instrument *Mathematical Meanings for Teaching secondary mathematics* (MMTsm). The MMTsm is focused on meanings because meanings, by their very nature, are implicative of action. If a teacher's meaning of equation is an image of two expressions separated by an equal sign together with procedures for operating on those images, we anticipate this will reveal itself in the teacher's utterances, actions, and in their expectations of students' learning. If a second teacher's meaning of an equation is a question about what elements of a domain make an open statement true, we likewise anticipate that this will reveal itself in instruction. We also anticipate that the two teachers' students will learn different mathematics from the two teachers' instruction.

Second, we ground the design of assessment items and methods for interpreting teachers' responses in a theory of meaning that is inspired by Piaget's genetic epistemology (Piaget & Garcia, 1991; Steffe, 1990; Thompson, 2013; Thompson, Carlson, Byerley, & Hatfield, in press). This theory provides a coherent system for addressing relationships among understandings, meanings, and ways of thinking (Thompson, et al., in press; Thompson & Harel, in preparation). In Thompson and Harel's system, an understanding is a cognitive state resulting from assimilating. A meaning is the space of implications of an understanding—actions, images, meanings, and ways of thinking that can come to mind readily as a result of the understanding.[2]

[2] The fact that "meaning" refers to meaning is not a problem. It is akin to computer science, where a procedure is defined as a set of instructions that invoke primitive actions and/or procedures.

When someone has an understanding created by assimilating to a scheme, the scheme is the meaning of the understanding. To have a way of thinking is to habitually employ certain meanings and ways of thinking when reasoning in certain contexts.

ASSESSING MATHEMATICAL MEANINGS

Assessment items are influenced highly by what assessment designers imagine they are assessing. To create an item that has the potential to reveal teachers' meanings requires one to focus on allowing teachers to express what comes to mind readily upon reading the item—without inadvertently leading teachers to think, "What do they want me to say?" That is, the item should emphasize teachers' interpretations of the item, which sometimes amounts to designing the item so that a teacher's response implies his or her interpretation.

In this short paper we describe our method for designing and scoring items to assess teachers' mathematical meanings that we hypothesized are important for teaching secondary mathematics. We then discuss one item that exemplifies our method.

Designing items to assess mathematical meanings

Other methods attempt to assess teachers' mathematical meanings using concept maps (Novak, 1990; Williams, 1998) or semantic networks (Branca, 1980; Leinhardt & Smith, 1985; Sowa, 2006). These methods are inadequate for our purpose, because concept maps tend to reflect teachers' theories of what they know, not their knowledge-in-use (their meanings), and semantic networks demand detailed information that cannot be gotten with an assessment instrument. To design an item that has the potential to reveal teachers' mathematical meanings, we begin with epistemic models of meanings teachers might hold with regard to the mathematical idea being targeted. The models can be inspired from many sources. Our primary sources are prior research, discussions in classes that we teach, and private interactions with students enrolled in a course.

Our overall method is this: (1) Create a draft item, interview teachers (in-service and pre-service) using the draft item. A panel of four mathematicians and six mathematics educators also reviewed draft items at multiple stages of item development. In interviews, we look for whether teachers interpret the item as being about what we intended. We also look for whether the item elicits the genre of responses we hoped (e.g., we do not want teachers to think that we simply want them to produce an answer as if to a routine question); (2) Revise the item; interview again if the revision is significant; (3) Administer the collection of items to a large sample of teachers. Analyse teachers' responses in terms of the thinking they reveal; (4) Retire unusable items; (5) Interview teachers regarding responses that are ambiguous with regard to meaning and it is important to settle the ambiguity; (6) Revise remaining items according to what we learned from teachers' responses, being always alert to opportunities to make multiple-choice options that teachers are likely to find appealing according to the meaning they hold;[3] (7) Administer the set of revised items to a large sample of teachers; (8) Devise scoring rubrics and training materials for scoring open-ended items; revise items only when absolutely necessary.

We have used this method to devise items in the areas of covariation, function (as model, as notation, or as object), proportionality, rate of change, structure (form), and frames of reference (coordinating two or more perspectives). Other papers submitted to PME from our project discuss particular items and their results.

We have several design strategies for revealing teachers' meanings. One is to ask teachers to describe what they would like their students to understand with respect to a mathematical word or phrase. We sometimes follow such an item with a separate question about how they would respond to a situation in which the meaning they just addressed is central. A second strategy is to devise an item that, we are confident, requires a particular meaning to answer cogently but ask it in a way that teachers who have less than cogent meanings can nevertheless interpret the item in a way that that fits the meanings they have. A third strategy is to ask teachers to deal with a representation in a way that reveals the meanings they have for the representational system by which it is made. A fourth strategy is to display an animation that embodies, from our perspective, an important mathematical meaning and ask questions about the animation that allow teachers to reveal the ways they understand what they are seeing.

An example item

Teachers can have many meanings for proportionality. One is an equality of ratios. Another important meaning is when two quantities are related so that changes in one are a multiple of changes in the other. This is the meaning of a linear relationship. We wanted to know what teachers understand at first blush about the implications that a linear relationship between two quantities has when one quantity's value changes.

Figure 2 shows the fourth version of an item designed to examine teachers' meanings for two quantities being related linearly. The first was open response, requesting that teachers place their own hash mark on the upper line and asking various questions about their possible use of the item in instruction. We arrived at the version in Figure 2 after teacher interviews, administration to 39 teachers, further interviews, and a second administration to 100 high school teachers.

Each mark in Figure 2 reflects a way of thinking we detected in prior interviews and administrations. In Figure 2:

- $\Delta x = (1/2)x$, so x changed by 50%—thus cx should change by 50%.

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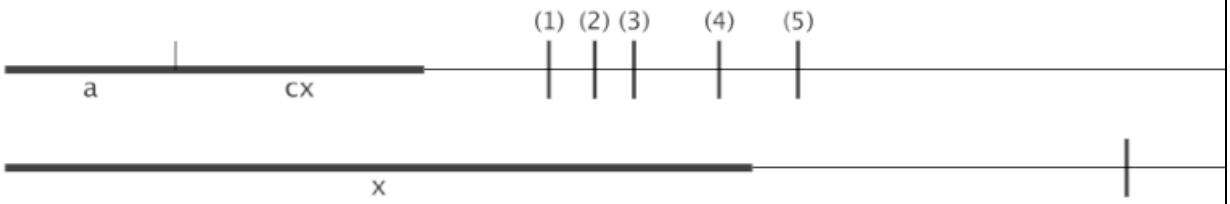
[3] This is the strategy employed by Carlson, Oehrtman, and Engelke (2010), who asked open-ended question of a large sample of students to see how they responded to questions involving precalculus ideas. They then interviewed students about their responses and constructed multiple-choice options that captured the spectrum of meanings they discerned from students' interviews.

- (1) is the hash for $(a + cx) + (1/2)cx$, which is the correct placement.
- (2) is the hash for adding a to $(a + cx)$.
- (3) is the hash for adding $(1/2)(a + cx)$ to $(a + cx)$.
- (4) is the hash for $a + (a + (1/3)(x + \Delta x))$; the teacher estimated that $c = 1/3$.
- (5) is the hash for $(a + cx) + \Delta x$.

a. In the diagram below, the lower bar represents a value of x. The upper bar represents a value of $cx + a$ for some numbers c and a.

The lower bar will be stretched to end at the hash mark on its line. Circle the hash mark at which you think the upper bar will end.

If you decide that none of the upper hash marks is correct, then place your own.



b. Explain how you decided upon the correct mark for the upper bar.

Figure 2. Fourth version of a proportionality item (© 2013 Arizona Board of Regents).

Version 4 of this item is much easier to score than the original, and we have an interpretation of each choice that is based in data. Selections 2 and 5 reflect additive meanings, and selections 1, 3, and 4 reflect multiplicative meanings (though 3 and 4 are mixed with additive). Teachers' explanations are important. Their explanations often confirmed our interpretation of their selection or clarified their interpretation of linear relationship—in rare cases teachers' visual estimates were inaccurate, leading them to select an option other than (1) even while reasoning appropriately, or to select (1) with inappropriate reasoning. But the limited number of options does focus teachers' interpretations.

Rubrics for scoring open response items

While multiple choice items are based in a theory of meaning for the particular idea being addressed, scoring them is relatively simple. Rubrics for scoring open response items that focus on revealing teachers' meanings are much more difficult to construct. These rubrics are typically 8-15 pages long. The length is due to needing to explain to scorers the item's purpose, rationale, and underlying theory—and we include many example responses to clarify the scoring strategy for that item. We have found that inter-rater scoring can be quite unreliable without including explanations and examples.

So far we have succeeded in generating 6 multiple choice versions out of 43 MMTsm items. Most items remain open response, largely because we could not avoid options that teachers find highly leading—"Oh, that makes sense! I would never have thought of that!"

DIFFERENCES BETWEEN ASSESSING STUDENTS' AND TEACHERS' MEANINGS

We began our project with the goal of devising items so that teachers' responses could be mapped to levels of meaning, where the levels reflect a theory of an epistemic learner's development of the meaning being investigated. We now believe that it is not possible to map teachers' responses to developmental levels of meaning for the mathematics they teach. Teachers are not first-time learners of the ideas we assess. Rather, in many instances they learned the ideas poorly, developing unproductive meanings, and then spent years learning to cope with mathematics instruction that they were unprepared to understand— developing ways to satisfy demands to perform without having a basis in meaning. As a result, our approach is to design rubrics to reflect levels of *productive* meaning—where "productive" is judged by a criterion of how useful this meaning would be for students' future mathematical learning were the teacher to convey it to them. In this endeavour we are attempting to use the method of construct maps and Wright maps (Wilson, 2005).

USING THIS METHODOLOGY IS NOT EASY

At the outset of our project, we anticipated that it would take 1.5 years to develop the MMTsm. The project is now in its third year. Figure 3 shows the numbers of revisions that items in the MMTsm underwent and the number of revisions that rubrics for open-response items underwent. We based each item revision on data—either interview or large-scale administration. We based each rubric revision on trial scoring and intensive discussion by members of our project. We did not anticipate the required effort to produce the MMTsm.

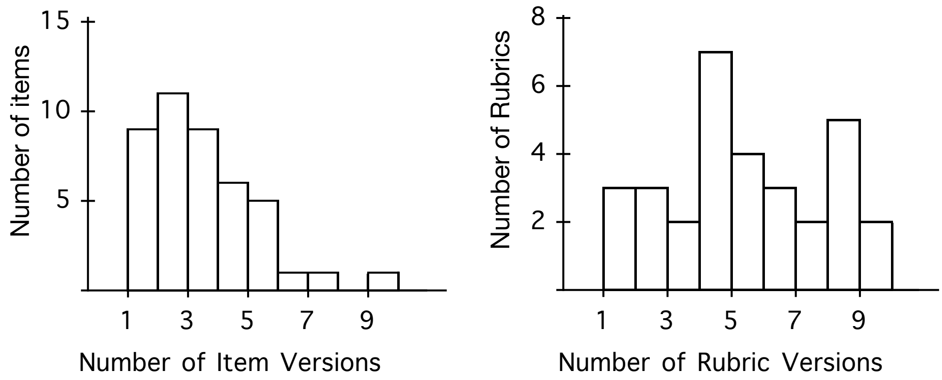


Figure 3. Left: Number of revisions for each MMTsm item (mean = 3)[4]. Right; number of revisions for each MMTsm open-response rubric (mean = 4.9)

FUTURE DIRECTIONS FOR ASSESSING TEACHERS' MEANINGS

An important question about the MMTsm arises naturally: Does it assess anything worthwhile? Put another way, do scores on it relate to qualities of mathematics

[4] This count does not include the 52 items that we eventually abandoned.

instruction? We envision two ways to address this question. The first is to learn from teachers what they think about the MMTsm's focus on mathematical meanings and to learn what they think about its implications for their instruction and for student learning. The second is to develop an observational protocol that focuses on meanings that are conveyed to students through teachers' instruction.

We held a summer workshop for 11 teachers who took the MMTsm to learn their thoughts about the MMTsm's focus on meanings and its implications for their instruction. While we cannot describe the workshop or its outcomes here, we are heartened that the MMTsm is addressing important issues of teaching and learning secondary mathematics. The development of an observation protocol will be the project's second phase, to begin in Spring 2014. In this effort we anticipate drawing inspiration from the Instructional Quality Assessment (IQA) (Boston & Smith, 2009; Boston & Wolf, 2006). While the IQA is not designed to address the conveyance of meaning, we anticipate that its focus on high quality communication and rich tasks will provide a good starting point.

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