TEACHERS' MEANINGS FOR FUNCTION NOTATION[1]

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This study explores teachers' meanings for function notation and function definitions. We focus on what teachers see as representing an output and as representing a function definition, and we analysed meanings and ways of thinking that teachers have according to what they see. Our analyses suggest that teachers who attend to a function's rule of assignment as representing its output or as constituting its definition operationalize function notation computationally, are unaware of inconsistent use of variables, and are oriented to find an explicit rule when asked to represent varying quantities. In this paper we also hypothesize that attention to the rule of assignment creates an inability to use function notation representationally.

INTRODUCTION

Students' understanding of functions and its importance has been a central issue of mathematics education research. Vinner & Dreyfus (1989) emphasized that making connections among various representations of function is important. DeMarois & Tall (1999) described that the facets of the function concept include function notation (using $f(x)$), the colloquial use of a function machine (input-output box), the standard symbolic (algebraic formula), numeric (table) and geometric (graph) facets. Among multiple facets of the function concept, function notation is one of several conventions to represent and name relationships between the values of two variables (Thompson, 2013b). However, there is little research on students' or teachers' understandings of function notation per se. In this study, we investigate teachers' meaning of function definitions that use function notation, and further how these associate with teachers' mathematical meanings of function concept.

THEORETICAL FRAMEWORK

According to Thompson (2013b), the convention for function definitions that use function notation includes the name of the function, the variable that represents a value at which to evaluate the function, and a rule that says how to determine the function's output given the input. In Figure 1:

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- *"u"* represents an input value—the value at which to evaluate the function.
- *"V(u)"* represents the function's output value when given the value of *u* as input.
- "=" means "is defined as".

$$\overbrace{\overbrace{\overbrace{V}^{\textit{Name}}\left(\overbrace{u}^{\textit{Input}}\right)}^{\textit{Output}} = \overbrace{u(13.76-2u)(16.42-2u)}^{\textit{Rule for how to produce an ouput}}}^{\textit{Function definition}}$$

Figure 1. Parts of a function definition (from Thompson, 2013b, p.4)

The left-hand-side represents the output of function *V*. This is because we can use "*V*(*u*)" to represent the output of function *V* regardless of whether we know its rule of assignment. In addition, the whole statement including the left-hand-side (output), "=", and the right-hand-side (rule for how to produce an output) constitutes the definition of *V* using function notation (as in Figure 1). A dictionary definition is a word together with its meaning. Similarly, a function definition is its name together with its rule of assignment.

We interpret teachers' responses in terms of a theory of meaning (Thompson, 2013a; Thompson, Carlson, Byerley, & Hatfield, in press). The theory employs a system for addressing issues of understanding, meaning, and ways of thinking developed by Thompson and Harel. According to Thompson et al., an understanding is a cognitive

Construct	Definition
Understanding (in the moment)	Cognitive state resulting from an assimilation
Meaning (in the moment)	The space of implications existing at the moment of understanding
Understanding (stable)	Cognitive state resulting from an assimilation to a scheme
Meaning (stable)	The space of implications that results from having assimilated to a scheme. The scheme is the meaning. What Harel previously called Way of Understanding
Way of Thinking	Habitual anticipation of specific meanings or ways of thinking in reasoning

Table 1. Definitions of understanding, meaning, and ways of thinking (from Thompson & Harel, in preparation)

state resulting from assimilating. When someone has an understanding created by assimilating to a scheme, the scheme is the meaning of the understanding. Table 1 shows the definitions that are key to our theory of meaning. Given that we are assessing teachers' mathematical meanings in light of this theory, questions that focus on what teachers can do (e.g. solving an equation) provide little insight into their meanings. Thus, the items that we use focus little on correct/incorrect answers and instead aim to characterize the meanings the teachers may possess.

RESEARCH METHODOLOGY

In the summer of 2013, we administered an assessment to 100 high school teachers in the Midwest and Southwest United States. The assessment contained items that have the potential to reveal teachers' meanings for various mathematical concepts such as function. A full description of the method to design items aimed at assessing mathematical meanings is beyond the scope of this paper; however, a full description is available (see Thompson & Draney, under review). We discuss five items pertaining to the function concept here.

Items

The item shown in Figure 2 is one of our attempts to uncover the meanings that teachers have for function notation. The intent of Item 1 is to reveal the teachers' understanding of the roles of each part of a function definition. In particular, it is significant that teachers understand the function's output to be represented by "$f(x)$",

Part A:

A function f over the real number is defined below. Circle what you like your students to think represents the function's output.

$$f(x) = x(11-2x)(8.5-2x)$$

Part B:

Here is a second function. Circle the function definition.

$$g(x) = x\sin e^{x}$$

Figure 2. Circling the output (Item 1A) and the function definition (Item 1B) (© 2013 Arizona Board of Regents).

as this is crucial to using functions and function notation to represent unknown quantities, especially when a function rule is not provided. In Part B, the function definition is the whole statement including the left-hand-side, "=", and the right-hand-side.

We believe that Item 2 (Figure 3) provide useful information on their meaning for function notation. Purposely we used different letters in the left-hand-side (x) and the right-hand-side (n) so as to see whether teachers notice variable inconsistency in James' definition. If they answer 45, then that is evidence that $f(x)$ is an idiomatic expression. In other words, the answer 45 would be a sign that he or she has the meaning that "$f(x)$", in its entirety, is a label for the formula that is on the right hand side of a function definition.

James, a student in an Algebra 2 class, defined a function *f* to model a situation involving the number of possible unique handshakes in a group of *n* people. He defined *f* as:

$$f(x) = \frac{n(n+1)}{2}$$

According to James' definition, what is $f(9)$?

Figure 3. James' definition (Item 2) (© 2013 Arizona Board of Regents).

The purpose of Item 3 (Figure 4) is to see the ways in which teachers operationalize function notation. This item assesses teachers' ability to evaluate functions defined with function notation. The highest-level response will be to evaluate each function at its input and use the output of other functions that appear in its definition. The next level of response will be when teachers substitute each function reference with the expression that defines it.

The functions *f*, *g*, and *h* are defined below

$$f(u) = u^2 - 1$$

$$g(s) = 1 + \frac{f(2s+1)}{2}$$

$$h(r) = g(r) - 1$$

What is $h(2)$? Show your work.

Figure 4. Finding $h(2)$ (Item 3) (© 2013 Arizona Board of Regents).

Figure 5 shows examples of evaluation and substitution in the context of Item 3. We hypothesize that if a teacher has a meaning for function notation so that $f(x)$ represents the output of the function *f*, then she use $g(2)$ and $f(5)$ as representations for the calculated values. In contrast, if a teacher's meanings for function notation include viewing the rule of assignment as the function's output, then that teacher would be pre-disposed to substituting each function's rule into the rule of *h*. In this sense, we view evaluation as higher than substitution as this approach requires a more flexible meaning for function notation.

$$\begin{aligned} h(2) &= g(2) - 1 \\ g(2) &= 1 + \frac{f(5)}{2} \\ f(5) &= 25 - 1 = 24 \\ g(2) &= 1 + \frac{24}{2} = 13 \\ h(2) &= 13 - 1 = 12 \end{aligned}$$

$$\begin{aligned} h(r) &= \left(1 + \frac{f(2r+1)}{2}\right) - 1 \\ &= \left(1 + \frac{(2r+1)^2 - 1}{2}\right) - 1 \\ h(2) &= \left(1 + \frac{(4+1)^2 - 1}{2}\right) - 1 = 13 - 1 = 12 \end{aligned}$$

Figure 5. Examples of evaluation thinking (left) and substitution thinking (right).

The purpose of both Item 4 and Item 5 (Figure 6) is to see whether teachers decide to use function notation. The focus of Item 4 is to see whether the teacher uses function notation on both sides of his or definition, to represent the ripple's radius, say $r(t)$, and the ripple's area, say $A(t)$. We focused on teachers' use of function notation on *both* sides of the function definition because we want to see whether the teacher is comfortable using function notation to represent varying quantities (even in the rule of another function definition, when the defining rule is unknown). Item 5 is an animated item where the right figure becomes larger and smaller at an irregular rate. Thus, it will not work to look for an explicit model to represent the right figure's varying lengths. The intent of Item 5 is to see whether teachers decide to use function notation to represent a quantity's value as co-varying with time. We are interested in whether teachers decide to use function notation, and if not, whether they at least try to capture lengths' variation with respect to the number of seconds elapsed. If a teacher proposed a model, we did not score it on its accuracy or viability. We cared only about how the teacher chose to represent lengths' variation with time.

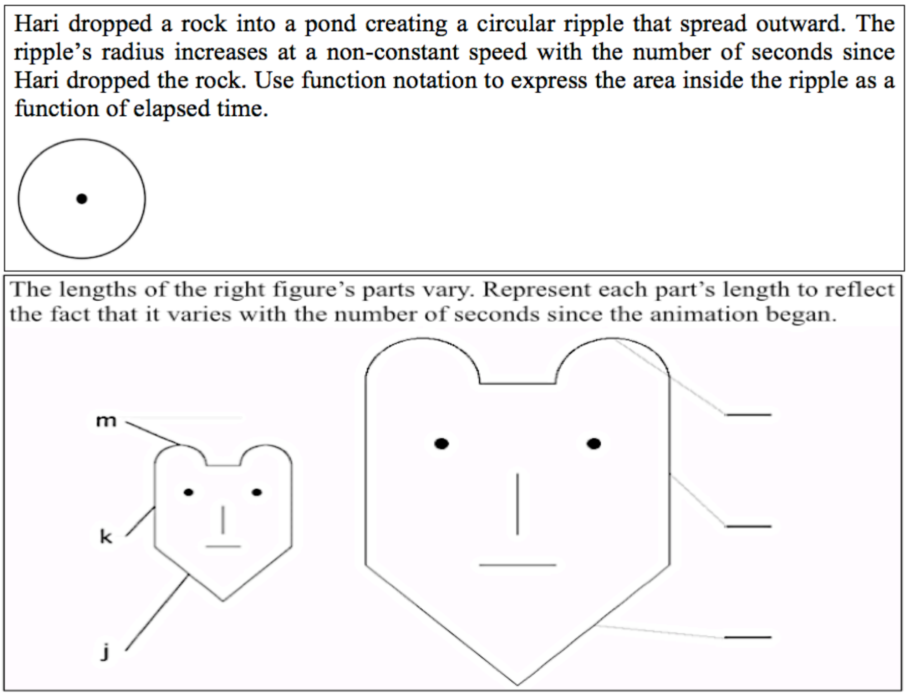

Figure 6. Hari's rock and Representing figures-Item 4 (upper) and Item 5 (lower) (© 2013 Arizona Board of Regents).

RESULTS AND ANALYSIS

Circling the output (Item 1A) vs. Finding *h*(2) (Item 3)

Table 2 shows the results of comparing teachers' responses to Item 1A (circling function output) and finding $h(2)$ in Item 3. Teachers who circled $f(x)$ as the output predominately evaluated functions using function notation. On the other hand, teachers who substituted each function reference with the expression that defines it tended to circle $x(11\text{-}2x)(8.5\text{-}2x)$ as the output. This result supports that teachers who found the rule for how to produce an output of the function h by substituting each function reference focused on a function's rule even though substitution approach takes a longer time than evaluation approach.

Portion Circled as Function Output	Strategy Used for Evaluating $h(2)$: Evaluate	Substitute	Function Name as Multiplication	Variable Inconsistency	Other	No Response	total
$f(x)$	30	1	0	2	3	1	37
$f(x)$ and $x(11\text{-}2x)(8.5\text{-}2x)$	6	2	0	0	0	0	8
$x(11\text{-}2x)(8.5\text{-}2x)$	20	7	0	6	4	0	37
Other	1	0	1	0	2	0	4
No Response	1	0	0	0	0	0	1
total	58	10	1	8	9	1	87

*Note: Only 87 of the 100 teachers saw item 3.

Table 2. Frequency two-way table for high school teachers' responses for circling the output of a function (Item 1A) vs. their strategy for finding $h(2)$ (Item 3).

Circling the function definition (Item 1B) vs. James' definition (Item 2)

In Table 2, 8 teachers responded to Item 3 with inconsistent variable usage much like what we purposely did in Item 2 with James's definition (Figure 3). The majority of these teachers focused on the rule of assignment. Additionally, we believe that selecting the right-hand-side (rule) of a function definition as the definition gives some indication that functions are about the rules. Focusing on just the rule could prevent teachers from noticing James' inconsistent use of variables in Item 2. Over half (34/63) of the teachers who circled only the rule as the definition of a function tended not to notice James' inconsistent use as shown in Table 3. Additionally, about two-thirds (13/21) of the teachers who circled the full definition did notice the inconsistency. This suggests that teachers who view the function rule as the function's definition are thinking more about rules for assigning values to a function than about representing a function's values. Such thinking could result in inattention to variable consistency.

Portion Circled as Function Definition	James' Variable Inconsistency: Noticed	Did not notice	No Response	total
$f(x)=x(11\text{-}2x)(8.5\text{-}2x)$	13	8	0	21
$x(11\text{-}2x)(8.5\text{-}2x)$	27	34	2	63
$f(x)$	0	7	0	7
Other	3	3	1	7
No Response	0	1	1	2
total	43	53	4	100

Table 3. Frequency two-way table for high school teachers' responses for circling function definition (Item 1B) vs. James' definition (Item 2).

Rule of assignment orientation (Item 1A & 1B) vs. Representing figures (Item 5)

Focusing on just the rule as an output and a function definition appears to lead teachers to an attempt to model situations with explicit rules. Explicit models are mathematical expressions that could be used to find the value of the varying lengths, for example *m*t* (*t*: the number of seconds) or |sin *t*|. Similarly, some teachers wrote a verbal description of the figure's behaviour such as "It decreases and then increases". In Table 4, 37 teachers circled $x(11\text{-}2x)(8.5\text{-}2x)$ as the output and 54 teachers circled this same expression as the function's definition; 17 teachers circled the expression in both cases. Thus, 74 teachers (out of 87) circled the rule as an output or a function definition. Only one of these teachers did not aim to find an explicit rule for the lengths of the animated figure's sides; this teacher used function notation. In other words, 73 of the 74 teachers who focused on the rule as a function definition aimed to find an explicit rule or did not use function notation when asked to represent the right figure's varying lengths.

Cells Contain:
Response to Item 5
*Uses Function Notation
*Verbal Description/Rule-Oriented Modeling
*No Function Notation/Modeling

Portion Circled as Function Definition (Item 1B)

Portion Circled as Function Output (Item 1A)	$f(x)$ = $x(11\text{-}2x)(8.5\text{-}2x)$	$x(11\text{-}2x)(8.5\text{-}2x)$	$f(x)$	Other	No Response	totals
	1	1	0	0	0	2
$f(x)$	3	15	0	1	0	19
	0	15	0	1	0	16
	0	0	0	0	0	0
$f(x)$ and $x(11\text{-}2x)(8.5\text{-}2x)$	2	4	0	0	0	6
	1	0	1	0	0	2
	0	0	0	0	0	0
$x(11\text{-}2x)(8.5\text{-}2x)$	7	11	3	0	0	21
	6	6	2	1	1	16
	0	0	0	0	0	0
Other	0	1	0	1	0	2
	0	1	0	1	0	2
	0	0	0	0	0	0
No Response	0	0	0	0	0	0
	0	0	0	0	1	1
	1	1	0	0	0	2
totals	12	31	3	2	0	48
	7	22	3	3	2	37

*Note: Only 87 of the 100 teachers saw Item 5.

Table 4. Frequency three-way table for high school teachers' responses for Item 1A vs. Item 1B, stratified by Item 5.

DISCUSSIONS

All three comparisons suggest that many teachers have a tendency to focus on the function's rule of assignment. We hypothesize that they, indeed, conceive of functions as being rules of assignment. These comparisons reveal some of the consequences of having that conception. To think of functions as a rule that assigns a value to an initial value via a formula emphasizes the process by which values are found and de-emphasizes representing the output of function. In the analysis of Items 1A & 1B vs. Item 5 (Table 4), our hypothesis helps to explain how the 73 teachers who circled the function's rule as the function's definition or as the function's output might have been thinking. Attempting to come up with explicit models or descriptions of observed behaviour is consistent with attempting to establish a rule of assignment.

Recall that items Item 4 and Item 5 (Figure 6) both aim to see whether teachers use function notation to represent varying quantities: teachers needed to use function notation on both sides of a function definition in Item 4 whereas Item 5 provided an opportunity for teachers to spontaneously use function notation to represent side lengths that co-vary with time. Teachers were specifically asked to use function notation in Item 4.

	Response to Item 5			
Use of Function Notation in Item 4	Function Notation to represent the lengths	Verbal Descriptions/Rule-Oriented Modeling	No Function Notation or Modeling	total
Represent values of Area and Radius Length	1	5	3	9
Represent values of Radius Length Only	1	6	2	9
To Label or Name	0	26	20	46
Not Used	0	11	12	23
total	2	48	37	87

Table 5. Frequency two-way table for level of high school teachers' responses to representing area (Item 4) vs. representing side length (Item 5).

The comparison between Item 4 and Item 5 (Table 5) shows that only two teachers among the 87 used function notation spontaneously in Item 5 and of those two only one used function notation on both sides of a function definition in Item 4. Under our hypothesis, representing the output of a function as $f(x)$ would be not be useful as $f(x)$ does not convey any information for how to assign values. Further research is needed to more fully examine our hypothesis.

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