The Common Core: Implications for Mathematical Practice

2013 Math and Science Partnership Learning Network Conference

Implementation: From Vision to Impact

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TERC
2067 Massachusetts Avenue
Cambridge Massachusetts 02140
617-547-0430

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A FEATURED TOPIC SESSION

Presenters:

Henry S. Kepner, Jr.
Professor Emeritus,
University of Wisconsin-Milwaukee

DeAnn Huinker
Professor, University of Wisconsin-Milwaukee

Al Cuoco
Distinguished Scholar/Advisor,
Education Development Center

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Cover: Group work focusing on Common Core Standards

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Principal Investigators
Joni_Falk@terc.edu
Brian_Drayton@terc.edu

Conference Documentation
Catherine McEver

About This Summary

This document focuses on one of the Featured Topic Sessions at the 2013 Math and Science Partnership Learning Network Conference (LNC). The intent is to provide a summary of presentation highlights as well participant discussion.

Abstracts for these presentations may be found at the 2013 LNC site on MSPnet (see URL at right). A video of this session as well as the original PowerPoint slide presentations are also available on MSPnet.

The LNC Online:

http://hub.mspnet.org/index.cfm/msp_conf_2013

Readers interested in pursuing any of the plenary session presentations from the 2013 LNC are encouraged to access MSPnet to find a document summarizing these sessions, full video recordings of the presentations, the original PowerPoint presentations, and detailed speaker biographies. All abstracts submitted for breakout sessions during the conference are also available on MSPnet.

A Brief History

Henry S. Kepner, Jr.
Professor Emeritus, University of Wisconsin-Milwaukee

Henry Kepner opens the session with a brief historical review of earlier thinking, work, and documents that led to the Common Core.

History Informing My Remarks

1980 An Agenda for Action

1989 Curriculum and Evaluation Standards for School Mathematics
- Addenda series

1991 Professional Teaching Standards

1993 Assessment Standards

2000 Principles and Standards for School Mathematics
- Navigations series

2001 NRC Adding It Up

2006 Curriculum Focal Points
- Grade Level/ Grade Band series

2009 Focus in High School Mathematics: Reasoning and Sense Making
- Statistics/Probability, Algebra, Geometry, Equity, Technology

2010 Common Core State Standards for Mathematics
- Standards Progressions Drafts

As with some of these earlier efforts, it may take a decade or more to interpret the Common Core, understand it, and come to some consensus regarding agreed-upon meanings, Kepner points out.

Drilling down into the more recent history of Common Core State Standards (CCSS), Kepner offers an overview of the development timeline.

Common Core State Standards

Following over 2 decades of standards development & refinement initiated by National Council of Teachers of Mathematics:

Spring 2009
National Governors Association (**NGA**) & Council of Chief State School Officers (**CCSSO**) agreed to develop a common core of state standards, starting in Mathematics and Language Arts.

Fall 2009
College and Career Readiness Math Standards

June 2, 2010
Common Core State Standards in Mathematics and English/Language Arts released.

June 2010
States started adopting the Mathematics and English/Language Arts standards.

When the NGA and the CCSSO decided to build a set of standards in 2009, there was no official sanction. They simply decided to put a team together and do it and made a conscious decision not to talk to the professionals at NCTM, AMS, and the like, Kepner relates. As with the standards, assessments are now being built with state input, and each state will make its own decision about what to do with those and whether to use them.

The process standards in *Principles and Standards for School Mathematics,* developed by NCTM in 2000, focus on what students should be

Summary:

The session examines challenges, interpretations, and substantive recommendations to engage students in the Standards for Mathematical Practice. Presenters will present a focus on supporting teachers to bring the “practice of mathematics” into their K-12 teaching of mathematics. Building on the early work leading to the NCTM Standards and decades of work on clarifying habits of mind, we will examine share professional development examples from our work with teachers. Recommendations will be discussed for keeping the Standards for Mathematical Practice at the forefront of Common Core implementation and assessment.

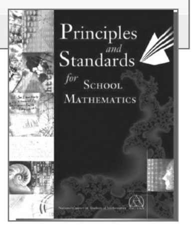
Milwaukee Mathematics Partnership
www.mmp.uwm.edu

COMMON CORE
STATE STANDARDS INITIATIVE
PREPARING AMERICA'S STUDENTS FOR COLLEGE & CAREER

Visit the Common Core State Standards Initiative website at:
http://www.corestandards.org/

CCSS Mathematical Practices

NCTM Process Standards	CCSS Mathematical Practices
Problem Solving	Make sense of problems and persevere in solving them. Use appropriate tools strategically
Reasoning and Proof	Reason abstractly and quantitatively. Critique the reasoning of others. Look for and express regularity in repeated reasoning
Communication	Construct viable arguments
Connections	Attend to precision. Look for and make use of structure
Representations	Model with mathematics.



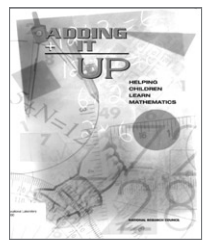
Henry Kepner

able to do. At left is one way of looking at these in relation to the CCSS Mathematical Practices and how they connect or overlap.

Adding It Up took a different approach, asking what it means for a student to be mathematically proficient.

Mathematical Proficiency
Adding It Up (NRC, 2001)

- **Conceptual understanding** – Comprehension of mathematical concepts, operations, and relations
- **Procedural fluency** – Skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- **Strategic competence** – Ability to formulate, represent, and solve mathematical problems
- **Adaptive reasoning** – Capacity for logical thought, reflection, explanation, and justification
- **Productive disposition** – Habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy



Kepner encourages participants to go back and look at some of these earlier documents, which present similar ideas but with a different slant or perspective.

The NCTM standards, he notes, represent a belief statement from a professional organization of members getting together and attempting to reach consensus. The NRC effort involved a broad view of authors looking at the same past research documents to identify factors that contribute to a student's mathematical proficiency .

- The NCTM Process Standards are a statement of belief about what is valued for students in mathematics.
- The NRC definition of mathematical proficiency is built on the research on children's mathematical knowledge.
- Habits of Mind (Cuoco) within profession.
- The Standards for Mathematical Practice were written into the standards agreed upon by the governors and their chief state school officers – a statement of law.

The standards present a new wrinkle, Kepner observes. All of the points above were voluntary. They helped to initiate conversation, but those who chose not to do them didn't. This led to problems regarding how much got implemented in schools and in teacher preparation programs, particularly in regard to the practices. With the Common Core Standards, those practices are built in, and when the governors say that people will be held accountable it is not just to the content but to the practices also. "We have to keep reminding people that those practices are part of that legal document," Kepner says. "Your state and your state superintendent have signed on saying that is part of what we expect to see from our kids."

Much of this is an equity issue, he observes. The courses that students take have a big impact on opportunities they then have for reasoning and sense making.

Equity

Mathematical reasoning and sense making must be evident in the mathematical experiences of *all* students.

- **Courses** students take have an impact on the opportunities that they have for reasoning and sense making.
- Students' **demographics** too often predict those opportunities.
- **Expectations**, beliefs, and biases have an impact on the mathematical learning opportunities provided for students.

Kepner closes with a quote from Henry Pollack, an applied mathematician and research director at Bell Labs. "This is a good statement to consider as we look at mathematical practice during this session," Kepner observes.

Learning Sequences and Pathways

"I believe the curriculum should be based in a fundamental way on two partial orderings, one of which is essentially supplied by the discipline and the other by society."

Henry Pollak
Mathematics Teacher (1977, pp.293-296).

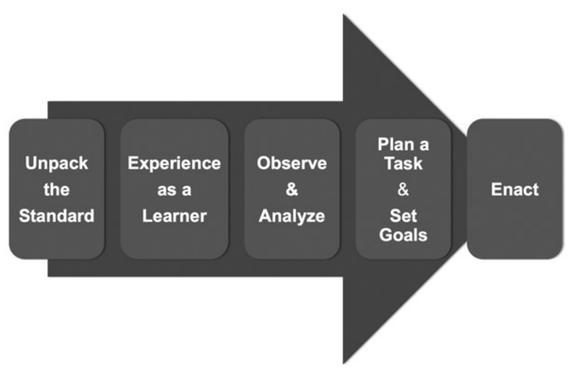
Engaging K-8 Teachers

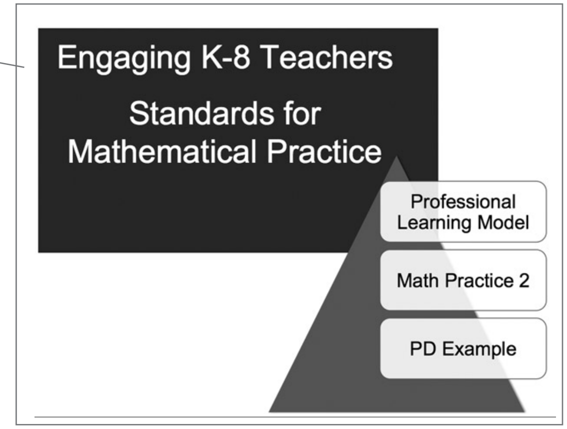
DeAnn Huinker
Professor, University of Wisconsin-Milwaukee

DeAnn Huinker engages session participants in reasoning and sense making, sharing work that she is doing with K-8 teachers. Huinker works with school districts to develop Common Core Leadership Teams that then help each district move further in their transition to the Common Core.

In thinking about how to engage teachers, Huinker explains, you begin by unpacking the standard and you then want them to go and enact the practices in their classrooms. Huinker utilizes some important intermediate steps to engage teachers on that path. First, teachers are involved in an activity so that they can experience as a learner what a practice might entail. Next is "observe and analyze," which involves getting a clear lens on a practice or content standards in relation to student learning and teacher practices in the classroom, and

Professional Learning Model





DeAnn Huinker

MP2. Reason abstractly and quantitatively.

Read Mathematical Practice 2.

Pay attention to key words or phrases that seem particularly important or that intrigue you.

Make a note of questions you have about particular aspects of the mathematical practice.

Highlight:
☆ Key ideas
? Questions

As a table group, discuss: To what extent do you think this practice is embedded in the daily work of teachers and students?

[See Appendix for full text of practices]

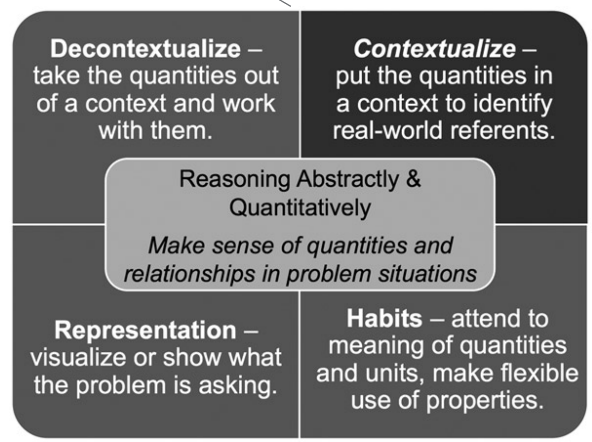
then watching videos and analyzing them. The same may be done with a written piece. Another strategy involves giving teachers some time to plan an instructional task or assessment task that targets a standard and a practice. "Not eight, not four, but one," Huinker explains. She then asks teachers, "What are you going to do with this task, or how are you going to develop it so that it really gives you some insight on kids' ability and proficiency on that task?" Huinker notes that she finds it disturbing when teachers watch a video and when asked what practices they observed, the common response is, "All of them." There is a need to get more focused, she emphasizes.

Huinker's process also involves having teachers generate a list of goals regarding a practice for a period of several weeks. They keep logs during this time and bring those back.

Huinker then engages session participants in an exercise.

After the exercise Huinker observes that as reflected in the table discussions, teachers are doing a lot of things around this practice, whether consciously or unconsciously, but there is a lot more that could be done to make it stronger and deeper and to help students.

In looking at this standard there are things that jump out at you, she notes, including "decontextualize" and "contextualize"—taking quantities out of a context and working with them, and putting them back into a context and working with them.



There is also "representation," the ability to visualize or show what is going on with a problem. Finally there are "habits," behaviors we want students to be developing in the classroom such as attending to the meaning of quantities and units.

Huinker then focuses on "contextualize," while noting that it plays into other areas as well. In working with teachers, practices are always connected to a content standard so that it is

Working groups

possible to see how both content and practices are being addressed, she explains.

Content Standard: Grade 6

Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

6.NS.1. **Interpret** and compute quotients of fractions, and **solve word problems** involving division of fractions by fractions, e.g., by using **visual fraction models** and equations to represent the problem.

Most teachers say they have never been asked to make a visual fraction model for division of fractions, Huinker reports, nor have they been asked to think about word problems very much. She then asks participants to attempt and discuss a task posed to teachers.

Experience

Pose a word problem for: $5 \div \frac{3}{4} = ?$

- **Contextualize the equation.**
- **Attend to the meaning of the quantities.**
- **Consider the units involved.**

After the exercise Huinker observes that it is sad that while preservice teachers may have been asked to do this (though not often) teachers in the field report that they have never been asked to do this before, and yet the standards are saying that teachers need to ask their students to do this. “In professional development we have to create opportunities for teachers to make sense of the mathematics, even some of that unlearning and unpacking,” Huinker says.

She cites a common measurement example she heard during the exercise: You have five cups of sugar, each batch of cookies requires ¾ cup of sugar, how many batches can you make? This gives us a context for thinking about how many three-fourths there are in five, she explains.

Another part of this content standard talked about visualizing, while the practice talked about coherent visualization of this problem. In order to do that you need the context, she points out. What would the equation at right look like if it were drawn out? Considering how to visualize and solve this problem with a diagram is very challenging for teachers and for their students, Huinker reports. In talking to teachers about contextualizing the problem she talks to them about contextualizing the solution as well, and attending to the meaning of the quantities. Huinker then gives session participants a few moments to draw their own sketches regarding the problem.

Huinker notes that an answer preservice teachers or students often come up with after

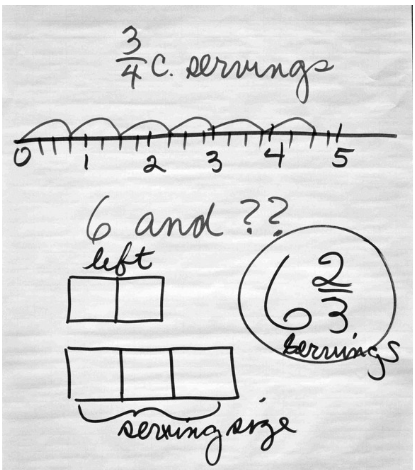
Working groups

Visualize and solve with a diagram:

$5 \div \frac{3}{4} = ?$

- **Contextualize the equation.**
- **Attend to the meaning of the quantities.**
- **Consider the units involved.**

$$5 \div \frac{3}{4} = ?$$



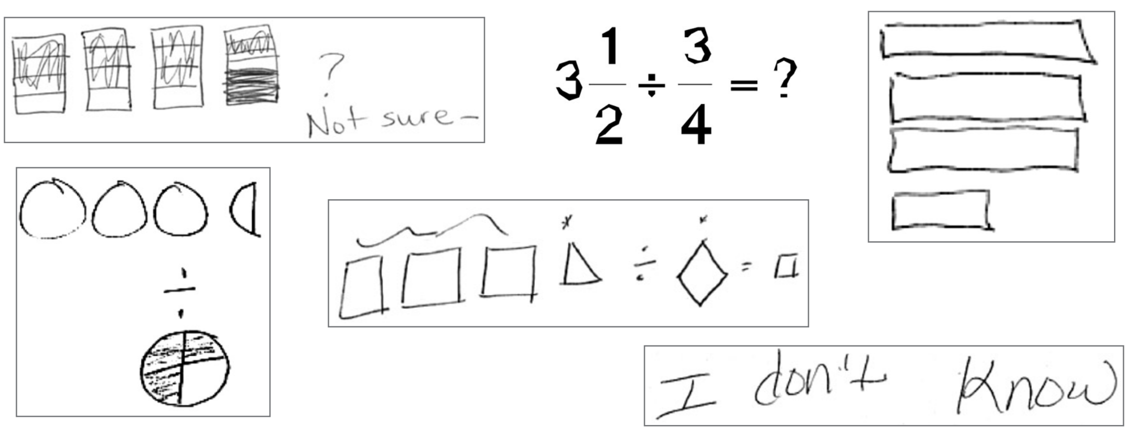
drawing a diagram is 6 ½. Then they do the algorithm and can't figure out whether they got the wrong answer with the algorithm or with the picture. It is very difficult to resolve what the 6 ⅔ answer means, Huinker notes. Even if they say it is 6 ⅔ servings, there is still the problem of that ⅔ remainder and what it means. "As we extend our understanding of division with whole numbers and fractions, we need to think about this," she says.

To illustrate this Huinker points to a similar problem below that she has used with teachers. Out of about 50 teachers in two different PD groups, less than five were able to do this, she reports. One person said there was a remainder, but said the remainder was one-half. Another was able to clearly explain that it was 4 ⅔ servings. Below are the initial types of responses Huinker received. Teachers struggle with what to draw and how to draw it. The number one answer, she reports, was "I don't know." This is a real struggle for them and these are fascinating to delve into, Huinker observes.

After a few months, out of 25 teachers Huinker worked with last fall, about 50% had a good explanation, resolving what to do with the leftovers. About 40% said they were still struggling with the idea.

Explain and show how to use a visual model to solve :

$$3\frac{1}{2} \div \frac{3}{4} = ?$$



Explain and show how to use a visual model to solve :

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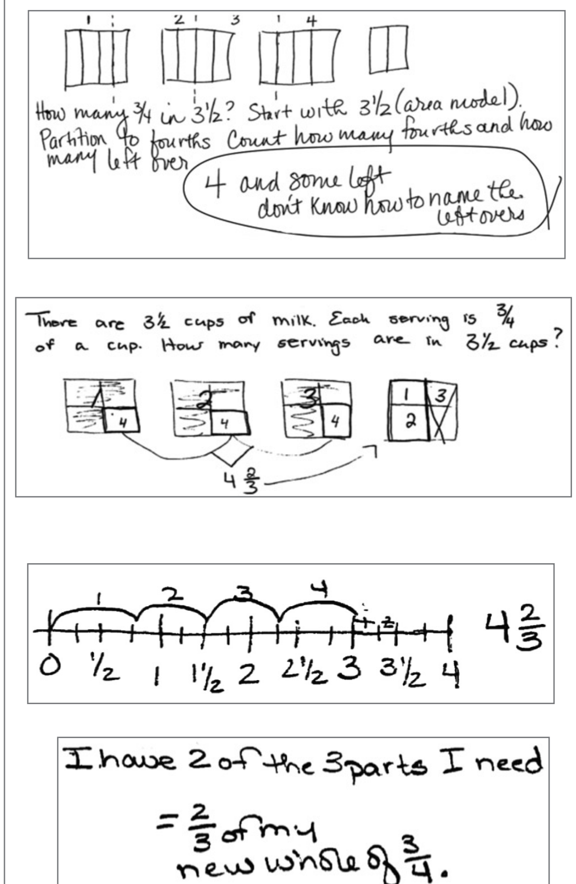

Another activity teachers are engaged in is thinking about developing tasks.

Plan Tasks

Instructional Task	• As grade levels, select a task. Modify the task to focus on a specific content standard and a math practice. • Bring lesson artifacts to discuss at our next project session.
Assessment Task	• In grade levels, develop an assessment task to check in on student proficiency with a content and practice standard. • Administer and bring student work samples to analyze with colleagues.

They are also engaged in setting goals. They set a few goals, work on those for a couple of weeks, and then come back and talk about them, Huinker explains.

Set Goals

Reason Abstractly and Quantitatively (MP2)

- Free write: Reflect on how much emphasis you give MP2 in your own classroom and identify specific examples.
- Small Groups: Discuss ways you could make MP2 more prominent in your teaching as an expectation of all students.
- Write down two goals as next steps to be more conscious of this MP prior to our next class.
- Keep a log for two weeks of classroom episodes, reflections, and insights.

In closing her portion of the presentation Huinker says, "I think it is important to give teachers a continuum of opportunities for thinking about these practices, allowing them to really dig in and delve by observing, by analyzing, by developing those plans and goals. Another thing I really believe in is that we have to stop doing all eight and focus on one, maybe two, at a time so that we can really bring this into our practice with the teachers."

Enacting the Mathematical Practices in classrooms is a journey for teachers ...

Summary

- Provide a continuum of opportunities to deepen knowledge and strengthen

Study → Experience → Observe → Plan → Goals

- Focus on a limited set, even one, Mathematical Practice at a time and work to become skilled at it.

High School Teaching: Standards, Practices, and Habits of Mind

Al Cuoco
Distinguished Scholar/Advisor,
Education Development Center

Al Cuoco proposes the following outline for talking about his work in and around Boston with high school teachers, noting that he will spend most of his time on item three, which revolves around a course developed for high school teachers about developing mathematical practice in their daily work.

Outline

1. The Practice of Mathematics
2. Standards for Mathematical Practice
3. Developing Mathematical Practice
4. One slide from PARCC
5. Parting Thoughts
6. More examples (if time)

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Al Cuoco

The Practice of Mathematics

What mathematicians most wanted and needed from me was to learn my ways of thinking, and not in fact to learn my proof of the geometrization conjecture for Haken manifolds.

—William Thurston
"On Proof and Progress in Mathematics."
Bulletin of the American Mathematical Society, 1994

The Practice of Mathematics

The widespread utility and effectiveness of mathematics come not just from mastering specific skills, topics, and techniques, but more importantly, from developing the ways of thinking—the habits of mind—used to create the results.

—Cuoco, Goldenberg, & Mark
"Habits of Mind: An Organizing Principle for High School Curricula." *The Journal of Mathematical Behavior*, 1996.

The Notion of Mathematical Practice

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise. . . . Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

—CCSS, 2010

The Practice of Mathematics

The distinctive way of thinking about mathematics that mathematicians use to do their work is something that people have been thinking about for a long time, Cuoco notes, citing the Thurston quote at left. He then turns to a quote from 1996, when he and others were involved in developing geometry curriculum with the idea of elevating the methods people used to the same level of importance as the results.

Cuoco points to another quote from a recent release by Hyman Bass, which represents an eloquent way of stating this assumption.

The Notion of Mathematical Practice

It will be helpful to name and (at least partially) specify some of the things—practices, dispositions, sensibilities, habits of mind—entailed in doing mathematics. . . . These are things that mathematicians typically do when they do mathematics. At the same time, most of these things, suitably interpreted or adapted, could apply usefully to elementary mathematics no less than to research.

—Hyman Bass
"A Vignette of Doing Mathematics." *The Montana Mathematics Enthusiast*, 2011.

Finally, he offers a quote from the Common Core itself.

Standards for Mathematical Practice

Turning to a list of the Standards for Mathematical Practice, Cuoco observes that there was celebration when these came out and the feeling that they were a long time coming.

Common Core: Mathematical Practices

Eight attributes of mathematical proficiency:

1. Make sense of complex problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Then two things started to happen, he says.

Two Emergent Problems

1. These standards have been treated in isolation from each other and from the content standards.
2. Many practitioners find the statements of 3, 7, and 8 obscure.

Commissioned by the MA Department of Elementary and Secondary Education, we developed a 45 hour course, *Developing Mathematical Practice* aimed at helping high school teachers implement these standards across the entire high school program.

First, the standards are being treated as separate entities rather than being seen as fluid and exemplars of the whole practice of mathematics itself. They were treated as separate from the content standards as well, and you can't teach a practice in a way that is divorced from the actual content, Cuoco points out. Another problem was that many practitioners found the statements in 3, 7, and 8 obscure.

The Educational Development Corporation (EDC) was commissioned by the Massachusetts Department of Education to develop a 45-minute course, Developing Mathematical Practice (DMP), around these mandated standards to help high school teachers implement them in their everyday teaching.

Developing Mathematical Practice

The principles of the course are outlined below. The attempt was to deal with the everyday content that teachers deal with in high school. Nothing exotic, Cuoco explains, but things like graphs, common equations, making geometric constructions. "It is easy to come up with very clever tasks that illustrate some of these practices," he observes, "but that is not what is going to be most useful to them. They need to integrate this into that bread-and-butter stuff they do every day."

While there was a focus on individual practices and individual themes, the overall focus was on the practice of mathematics, Cuoco stresses. Those familiar with high school know that there are different methods for solving word problems, graphing lines, and so on, and they are all disconnected. "They don't have to be that way," he points out. "The main use for Standards for Mathematical Practice in high school is to bring some coherence to this zoo of stuff. You can then fuse very general purpose tools that serve you well in algebra and geometry and calculus, all the way across."

The course has been run twice and is now a regular offering of the State of Massachusetts and of EDC, Cuoco reports. "What is most grati-

The DMP Course

The course is designed to show how the standards for mathematical practice

- enhance the teaching and learning of standard content by making it
  - more understandable to students
  - easier to teach
  - more satisfying to all involved
- form a web of practices, all connected in inseparable ways
- bring coherence and parsimony to the zoo of topics and methods in a standard high school program

The course is now a regular offering of ESE and EDC, conducted around the country and taught by teams of teachers, many of them members of the Focus on Mathematics community.

The DMP Course

Day 1: Getting Started

- Maximizing area for fixed perimeter
- SSS and area
- Minimizing paths
- Overview of the standards for mathematical practice

Day 2: Abstracting Generality from Repeated Reasoning

- Word problems
- Graphing
- Heron's formula
- Modeling functions
- Lines of best fit

Day 3: Structure

- Integers and Polynomials
- Fitting functions to tables
- Factoring
- Quadratics
- Monthly payments
- Probability distributions
- Complex numbers

Day 4: Variable Arguments

- Area and dissections
- Congruence → area → similarity
- Critiquing proofs
- Reasoning about irrational numbers
- Extending definitions
- Critiquing the reasoning of others
- Mathematical induction

Day 5: Implications for High School

- Designing activities for students
- The PARCC Content Frameworks
- End with some lovely mathematics

fying is that in Massachusetts and around the country, almost all of these sections are being taught by teachers that we have worked with, in some cases for ten years. The teachers work in pairs at sites in Massachusetts and around the country."

Referring to the DMP course outline, Cuoco notes that the middle three days focus on particular standards, but in ways that are inseparable from the others. The themes are the Standards for Mathematical Practice, but what is used to get at are things that are recognizable to most high school teachers, he explains. Within each day's content the course emphasizes how elements, which are treated separately now in high school, can be brought together by the employment of this way of thinking.

On the final day, because Massachusetts in a PARCC state, there is a focus on the PARCC Content Framework, as well as designing activities for students and "some lovely mathematics."

Cuoco then offers an example from algebra. He notes that the "dreaded algebra word problem," the butt of many jokes, is very difficult for many kids. The folklore says that the difficulty lies in things like reading level and context. Kids have a hard time reading the problems and the context doesn't make any sense.

Example: Building Equations
Word Problems

The dreaded algebra word problem

Mary drives from Boston to Chicago, and she travels at an average rate of 60 MPH on the way down and 50 MPH on the way back. If the total trip takes 36 hours, how far is Boston from Chicago?

Why is this so difficult for students?

- Reading level
- Context

Cuoco asks participants to compare the two different word problems below. Over and over again, he says, teachers say that many of the kids who can solve the second problem and understand its meaning and context cannot solve the first one. Most teachers would agree that

But there must be more to it. Compare. . .

Mary drives from Boston to Chicago, and she travels at an average rate of 60 MPH on the way down and 50 MPH on the way back. If the total trip takes 36 hours, how far is Boston from Chicago?

with

Mary drives from Boston to Chicago, and she travels at an average rate of 60 MPH on the way down and 50 MPH on the way back. If Boston is 1000 miles from Chicago, how long did the trip take?

"The difficulty lies in setting up the equation, not solving it."

the difficulty is in setting up the problem, not the math, Cuoco observes.

This is where the habit of abstracting regularity from numerical examples comes into play, he explains.

Here's where abstracting from numericals can help

- Take a guess, say 1200 miles.
- Check it:
  - $\frac{1200}{60} = 20$
  - $\frac{1200}{50} = 24$
  - $20 + 24 \neq 36$
- That wasn't right, but that's okay—just keep track of your steps.
- Take another guess, say 1000, and check it: $\dfrac{1000}{60} + \dfrac{1000}{50} \stackrel{?}{=} 36$

Capitalize on the fact that the bottom problem is easier for many people to solve than the top. Take a guess and say 1,200 miles. The object is not to get the right answer, Cuoco notes, but to get the equation. Then you check it. If you watch kids do this, you see that their work is all over the place, he notes. So they take another guess and after three or four times, you end up with the kind of thing shown in the final bullet point above.

"Eventually you come up with what we call a generic 'guess checker,'" Cuoco explains. "What is going on here is that they are abstracting from the actual calculations, and what they are abstracting is not a number it is a process. "That is what this standard is all about, abstracting regularities from repeated actions."

- Keep it up, until you get a "guess checker"

$$\frac{\text{guess}}{60} + \frac{\text{guess}}{50} \stackrel{?}{=} 36$$

- The equation is

$$\frac{x}{60} + \frac{x}{50} = 36$$

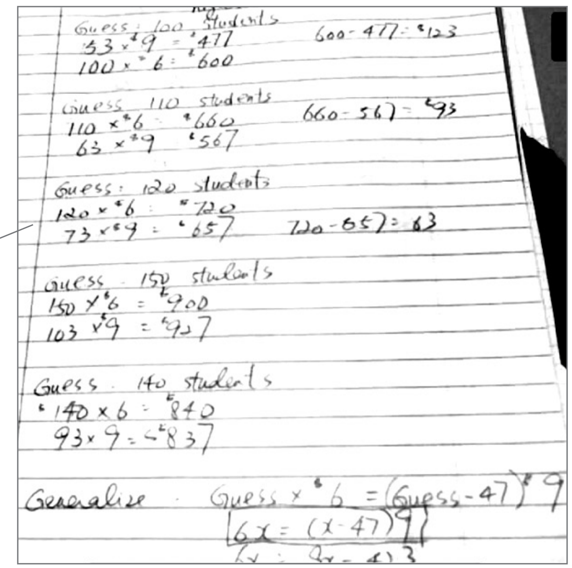
Below is a problem from the course. "In the old days there would be all kinds of ways to set up boxes and devices to do this," Cuoco says, "but what we like people to be able to do is play with it, take a guess, see what happens."

Example: Building Equations
from the course

Mr. Brennan is looking at a theater marquis:

He says "Isn't that great. The theater usually charges $9.00 per person. On Student Night, I can bring 47 more students for the same cost." How many students can Mr. Brennan bring to the theater on Student Night?

Here is the work of a high school teacher from the course. First he tries 100, which didn't work, then he tries 110, which didn't work. What is very important is that this is not guess and check, Cuoco notes, it is guess, check, and come up with an equation. Finally the teacher generalizes the whole thing. "That is what we hammer at," says Cuoco, "and that practice comes up all over the place."



Other Examples Where this Habit is Useful

- Finding lines of best fit
- Building expressions ("three less than a number")
- Geometric optimization
- Fitting functions to tables of data
- Deriving the quadratic formula
- Deriving Heron's formula
- Using recursive definitions in a CAS or spreadsheet

Cuoco proceeds to a problem using factoring, and shows an example from a popular 1980 text. It is possible, he notes, to write 24 problems that orchestrate your way through a theme, but this doesn't do that. "These are sort of random." In fact, he comments, you could write a computer program to do this if you were allowed to do so.

Example: Factoring
(from a popular text ~ 1980)

"Factoring Pattern for $x^2 + bx + c$, c Negative"

Factor. Check by multiplying factors. If the polynomial is not factorable, write "prime."

1. $a^2 + 4a - 5$	**2.** $x^2 - 2x - 3$	**3.** $y^2 - 5y - 6$
4. $b^2 + 2b - 15$	**5.** $c^2 - 11c - 10$	**6.** $r^2 - 16r - 28$
7. $x^2 - 6x - 18$	**8.** $y^2 - 10c - 24$	**9.** $a^2 + 2a - 35$
10. $k^2 - 2k - 20$	**11.** $z^2 + 5z - 36$	**12.** $r^2 - 3r - 40$
13. $p^2 - 4p - 21$	**14.** $a^2 + 3a - 54$	**15.** $y^2 - 5y - 30$
16. $z^2 - z - 72$	**17.** $a^2 - ab - 30b^2$	**18.** $k^2 - 11kd - 60d$
19. $p^2 - 5pq - 50q^2$	**20.** $a^2 - 4ab - 77b^2$	**21.** $y^2 - 2yz - 3z^2$
22. $s^2 + 14st - 72t^2$	**23.** $x^2 - 9xy - 22y^2$	**24.** $p^2 - pq - 72q^2$

Below is an example from 2010, 30 years later. The instructions tell you to follow six steps. "You couldn't make this up," he says amidst general groans from session participants. "This is why factoring gets a bad name." The Common Core has a lot about seeing structure and expressions, he notes, but that has nothing to do with this example, which consists of nonsense that never gets used again outside of high school.

Example: Factoring
From a published text (2010)

To factor a trinomial of the form $ax^2 + bx + c$ where $a > 0$, follow these steps:

A	B	C
F	H	D
G	I	E

1. Identify the values of a, b, c. Put a in Box A and c in Box B. Put the product of a and c in Box C.
2. List the factors of the number from Box C and identify the pair whose sum is b. Put the two factors you find in Box D and E.
3. Find the greatest common factor of Boxes A [sic] and E and put it in box G.
4. In Box F, place the number you multiply by Box G to get Box A.
5. In Box H, place the number you multiply by Box F to get Box D.
6. In Box I, place the number you multiply by Box G to get Box E.

Solution: The binomial factors whose product gives the trinomial are $(Fx + I)(Gx + H)$.

Cuoco then offers another example, reporting that most teachers say they have no problem with this and the kids understand it. These are called "sum-product" problems.

Example: Factoring Using the Structure of Expressions

Factoring monic quadratics:
"Sum-Product" problems

$$x^2 + 14x + 48$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

so. . .
Find two numbers whose sum is 14 and whose product is 48.

$$(x + 6)(x + 8)$$

But in the one below, problems arise. It takes

Example: Factoring Using the Structure of Expressions

What about this one?

$$49x^2 + 35x + 6$$

Here's where seeking and exploiting structure can help.

$$\begin{aligned} 49x^2 + 35x + 6 &= (7x)^2 + 5(7x) + 6 \\ &= \clubsuit^2 + 5\clubsuit + 6 \\ &= (\clubsuit + 3)(\clubsuit + 2) \\ &= (7x + 3)(7x + 2) \end{aligned}$$

deliberate practice to be able to develop the practice of being able to see how that seeking and exploiting structure can help, Cuoco observes. This example is rigged for the course because it is worked out so that the structure is just under the surface, he acknowledges.

Cuoco then presents another example.

Example: Factoring Using the Structure of Expressions

What about this one?

$$6x^2 + 31x + 35$$

$$\begin{aligned} 6(6x^2 + 31x + 35) &= (6x)^2 + 31(6x) + 210 \\ &= \clubsuit^2 + 31\clubsuit + 210 \\ &= (\clubsuit + 21)(\clubsuit + 10) \\ &= (6x + 21)(6x + 10) \\ &= 3(2x + 7) \cdot 2(3x + 5) \\ &= 6(2x + 7)(3x + 5) \text{ so . . .} \end{aligned}$$

$$6(6x^2 + 31x + 35) = 6(2x + 7)(3x + 5)$$

You can't do this one, he states. Why? One of the teachers in the course said to remember that you owe it a six.

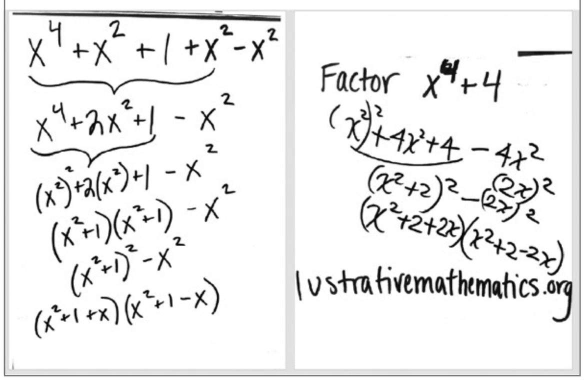
$$\cancel{6}(6x2 + 31x + 35) = \cancel{6}(2x + 7)(3x + 5)$$

$$(6x2 + 31x + 35) = (2x + 7)(3x + 5)$$

In this case you are introducing structure that wasn't there in the first place, Cuoco explains, and then you are just exploiting the structure and making use of it. He notes that this can save a week of time in algebra class because you don't have to worry about all kinds of combinations with this method. "We are involved with all kinds of special purpose methods that don't work. This works for general polynomics," he states. "It's a way to change the variables to make any polynomic you want."

Cuoco offers examples of teachers' work using this method showing a variety of applications.

Example: Factoring Using the Structure of Expressions

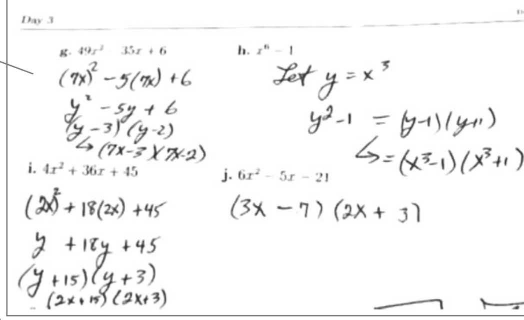


He then points to other examples where the habit of chunking is useful, treating pieces of an expression as a single thing.

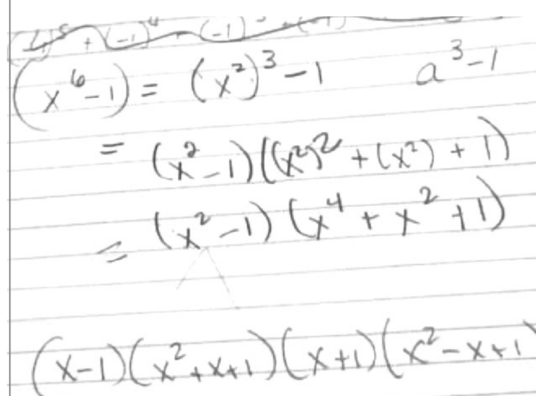
Other Examples Where Chunking is Useful

- Completing the square and removing terms
- Solving trig equations
- Analyzing conics and other curves
- All over calculus
- Interpreting results from a computer algebra system

Example: Factoring Using the Structure of Expressions



Example: Factoring Using the Structure of Expressions



Participant Q & A

Q: Structure isn't limited to algebra is it?

A: Correct, but I was asked to speak about algebra. There is also structure in geometry, where you are making the structure apparent.

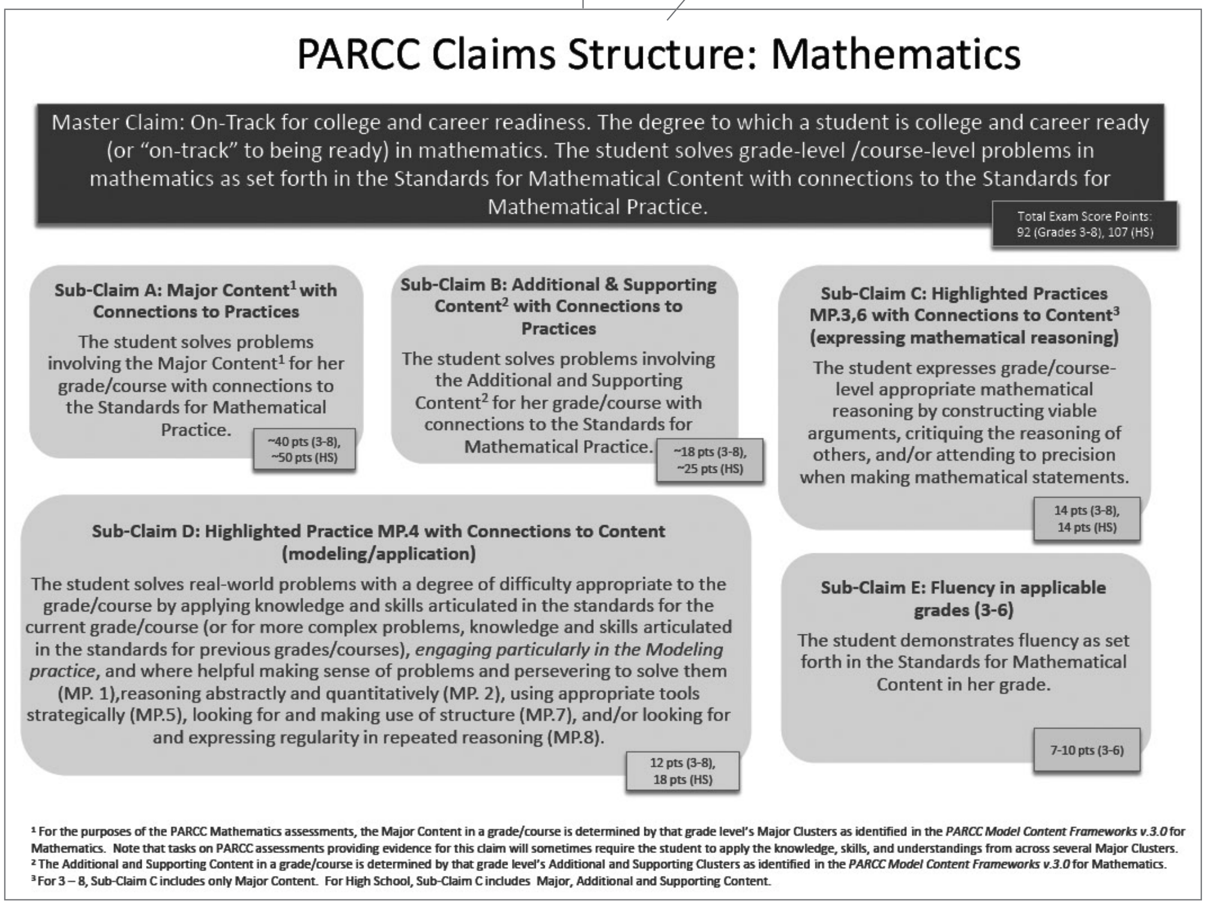
Thanks

acuoco@edc.org

http://mpi.edc.org/dmpsyllabus

One Slide from PARCC

Cuoco moves on to a slide from PARCC. Noting that it is not particularly readable, he explains that the point is that content is getting 50 points on the high school exam, but Standards for Mathematical Practice make up the rest. “They’re making serious attempts to find ways to assess practice,” he observes.



Parting Thoughts

Cuoco leaves participants with the following parting thoughts.

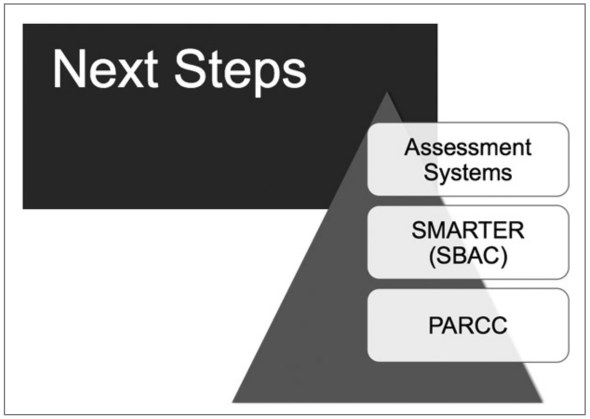
Conclusions

- There is a practice of mathematics, just as there is a practice of medicine or teaching.
- These Standards for Mathematical Practice capture some essential features of this practice.
- Elevating the habits of mind used to create results to the same level of importance as the results themselves can go a long way to connect school mathematics to the real thing.
- But the Standards for Mathematical Practice will be trivialized if they are not integrated into the Standards for Mathematical Content.

What Happens Next?

Henry S. Kepner, Jr.
Professor Emeritus,
University of Wisconsin-Milwaukee

Henry Kepner addresses the question of what happens next, noting that the Smarter Balanced Assessment Consortium (SBAC) says essentially the same thing PARCC does, though the focus may be a little different.



It will take good exemplars of the tasks and good exemplars of the assessment, to let people know what is going to be expected and to let teachers know what is going to be expected of their students. Kepner observes, adding that the hope is that good exemplars will surface from MSPs and at this LNC.

Development of assessment systems is being funded by the Department of Education and about half the states have signed on to PARCC and half are in the SBAC. Kepner emphasizes the point he made in his opening remarks that no one knows as yet what will be done with

Next Steps in Assessing Common Core Standards

- **The standards will make sense only when we have instructional and assessment exemplars to use and analyze** —the operational definitions!
- **Monitor the assessment developments**. DoE funded Consortia to develop assessment systems for use by 2014-15
  - Partnership for the Assessment of Readiness for College & Careers (PARCC includes NY, Achieve)
  - SMARTER Balanced Assessment Consortium

what is developed. However, he adds, this is an attempt to do it right. There are two groups working on this and there will be two tools in competition with each other, which is powerful, and there are things that each can learn from the other.

Kepner points to claims one and two from SBAC, observing that in the SBAC process, the dialog with the states regarding what will included in assessment involves give and take as individual items are negotiated. He notes that claim one, Concepts and Procedures, is something those involved think they do pretty well through multiple choice.

Regarding claim two, Problem Solving, he explains that some of those problems will be in context and some will be mathematical. To make sure the “math wars” people don’t get into this, there will be some of both. Some will involve a short answer, others will require the students to show some evidence of their thinking. Assessment can be done technologi-

SBAC Claims 1 & 2

1. **Concepts and Procedures**
   Students can explain and apply mathematical concepts and interpret & carry out mathematical procedures with precision and fluency.
2. **Problem Solving**
   Students can solve a range of complex well-posed problems in pure and applied mathematics, making productive use of knowledge and problem solving strategies.

SBAC Claim 3

3. Communication & Reasoning
Students can clearly and precisely construct viable arguments to support their own reasoning and to critique the reasoning of others.

Mathematical Practice 3

Understand & use stated assumptions, definitions, and previously established results in constructing arguments .

Make conjectures and build a logical progression of statements to explore the truth of their conjectures.

Justify their conclusions; communicate them to others.

Listen to or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

cally using something like an iPad to write their work, which it will then be possible to follow. Technologically, the tool can show the rubric scoring of the work as the student develops it. In a sense, Kepner explains, you could follow the sequence of the student's recorded reasoning.

Moving on to claim three, Kepner explains that student work in relation to this claim might involve the student looking at a sample of another student's work, determining whether it is correct or not, and explaining why. He points out how closely this parallels mathematical practice three.

Student work in relation to SBAC claim four might involve something simple or might take several days and be done in cohort groups. This may or may not be included in the first assessments, Kepner notes. They might not be ready and the states might not be ready for this as a high-stakes assessment tool.

SBAC Claim 4

4. Modeling & Data Analysis
Students can analyze complex, real-world scenarios and can construct and use mathematical models to interpret and solve problems.

Kepner concludes by posing a dilemma for session participants to consider. PARCC and SBAC are two large consortia that have a lot of money to do what the states want them to do: build assessments. "They may not last beyond 2015," Kepner points out. "Where is the money to keep them going? They are not in-place institutions. They are doing great things and we all have the opportunity to contribute, but they are moving into a vendors' market for assessment, where ETS, the College Board, ACT, and others already have established operations."

- Participate as the PARCC and SBAC consortia enter a vendors' market:
- ETS/College Board
- ACT
- Pearson
- Others

Governors and state superintendents are going to make decisions for their own states, he notes. "I think almost all of the governors and state superintendents who were in office when the Common Core was drafted are gone," he states. "We have a different audience there and they have different priorities"

Kepner relates the example of Wisconsin, an SBAC state, where the deputy superintendent serves on the seven-member SBAC executive board that serves as a driving force for the 24 states involved. Some months ago the state superintendent announced that SBAC tests would be used up through grade 8, and then high school would use ACTs. Neither Kepner nor others were aware this was happening. The decision was made at the superintendent's office.

Each session participant faces a potentially similar scenario in their own states. It is important to let decision makers know that those engaged in this work want these practices to be part of the assessment. "It is scary because this could disappear," Kepner says. "Part of it is time for the testing, part of it is money for the testing and the scoring. Those are all issues we have to be on top of."

Questions and Answers

Data from New Assessments

- I've been very concerned about the data that will eventually be produced by PARCC or the SBAC consortia tests that will eventually be administered. Will that data be available to research groups? • Session Participant

- Philosophically the answer is yes. However, the data is each state's data, so each state could make different decisions about access to data. • Henry Kepner

- If the states own the data separately, a question I worry about is whether there will then be a common format for it. • Session Participant

- They're working to provide a common format, but legally the governors wouldn't go anywhere unless they were given the authority because they're worried how it will be released to their publics. • Henry Kepner

- How would you find out more about who is thinking about that? • Session Participant

- At both PARCC and SBAC, each has a whole working group on those issues. Go onto the websites to get more information. There are serious teams looking at this, trying to negotiate what different states will accept, how to get access to data, and so on. It's very much a work in progress. The goal from the upper level was to make it as viable as possible to get at it all, but there may be local restrictions. • Henry Kepner

www.parcconline.org
www.smarterbalanced.org

The Challenge of Different Curriculum Trajectories

- The Common Core represents a significantly different trajectory and in particular I have a concern about high school geometry. Is there an existing curriculum that comes even close to aligning with what the Common Core is trying to get at? • Session Participant

- The pressure is on all of us to build something that uses and integrates transformations. At the high school level I would encourage you to go back and look at Arthur Coxford's *Geometry: a transformation approach*, written in 1971. All of us have to back up and rethink to respond to that in a timely manner, and there

Habits of Mind and NCTM

- Regarding what Al Cuoco is doing and what NCTM does around reasoning and sense making, is there any connection there or are they paying attention to each other? • Session Participant

- There is no formal connection, but it is pretty much the same. • Al Cuoco

Session participants

are bits and pieces out there that already exist that we can draw on. • **Henry Kepner**

- I'm not going to talk about the curriculum itself because we have a curriculum at EDC. But there is this notion: congruence → area → similarity. I think congruence and similarity are essentially different things in the minds of the kids. Congruence is a much more concrete thing; you can pick things up and move them around and you can eventually use the transformation approach as a pathstone for what they develop experientially around congruence.

  Similarity is a different thing because it depends on this real number. There is a scale factor that has to be there. There, I think it makes sense to approach similarity through dilations, so the results come from the properties of the transformations. With congruence I think it goes the other way around. The properties of the transformations come from the kids' actual physical manipulations of the things. • **Al Cuoco**

Appendix

Standards for Mathematical Practice

COMMON CORE
STATE STANDARDS INITIATIVE
PREPARING AMERICA'S STUDENTS FOR COLLEGE & CAREER

COMMON CORE
STATE STANDARDS

Standards for Mathematical Practice

1. **Make sense of problems and persevere in solving them.**
2. **Reason abstractly and quantitatively.**
3. **Construct viable arguments and critique the reasoning of others.**
4. **Model with mathematics.**
5. **Use appropriate tools strategically.**
6. **Attend to precision.**
7. **Look for and make use of structure.**
8. **Look for and express regularity in repeated reasoning.**

Milwaukee Mathematics Partnership

Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1. Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y – 2)/(x – 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x – 1)(x + 1)$, $(x – 1)(x^2 + x + 1)$, and $(x – 1)(x^3 + x2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.