Mathematical and Computational Tools to Observe Kepler's Laws of Orbital Motion

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This paper offers a new way to prove Kepler's laws using mathematical, computational, and visualization tools. We hope that it can be utilized in math and science courses. Our work has historical significance. As 9th graders, we had not taken any physics courses and we were not fully knowledgeable about laws of universe that govern planetary motion. That is not different from the situation of Kepler; as no one quite knew how gravitational forces worked until Newton came. Kepler had access to data compiled by Tycho Brahe and he looked for patterns [1]. We had access to modern tools and we looked for miracles! We learned how to transfer visuals images and data from simulation software (Interactive Physics) to geometrical software (Geometer's Sketchpad) to measure angles, distance, and areas of triangles needed for the proofs. Through Excel and simple algebraic steps (new = old + change), we learned how to manually construct our own simulations as an alternative way to get data needed for the proofs. To compute the "change" we needed some knowledge of the gravitational force that governs the orbital motion. While it was initially frustrating to learn new tools, demonstrations by our science teacher, Sean Metz, via a SmartBoard, eased this process. Realizing what Kepler would have done if he had such tools; we quickly learned to appreciate the opportunity in our hands. In the end, we did not make a discovery in physics, but we certainly discovered that physics was not a threatening or boring subject. We also discovered the role of mathematics in physics. The foreboding nature of complicated physics was abolished and we all looked forward to taking physics classes.

This paper is based on our experience in a math and science competition conducted by SUNY Brockport. Our project received the First Place and Best Paper awards. We are grateful to our teachers, the National Science Foundation, and the Computational Math, Science, and Technology Institute (www.brockport.edu/cmst).

Kepler's Laws [2]**:**

1. Orbit of a planet around the Sun is an ellipse
2. Planets sweep out equal areas in equal times
3. The square of the period (time to make a round) is proportional to the cube of orbit's semi-major axes.

Collecting Data with Interactive Physics

Here, we describe how Interactive Physics (IP) can be used to extract data on planetary motion to examine validity of Kepler's laws. IP is a commercial product, but demo versions are available at *www.interactivephysics.com*. Our school has a site license for it, but for those that do not have access, we will also illustrate below how the same data can be extracted from Excel using mathematical formulas and Newton's laws.

IP has a two-dimensional graphical interface. There is no need to remember laws of physics. One can even change the laws and their intensity through a drop-down menu. Objects of different shapes (circular, sphere, square) and materials (wood, steel, etc) can be created and tracked (via velocity and location graphs). Buttons can be created for control variables. Units can be scaled down to fit the window. Object properties can be viewed and edited easily.

To model the solar system through IP, one needs the solar data listed below [3]. To observe Kepler's 1st and 2nd laws, it would be adequate to construct only one planet around the Sun.

Planets	Mean Orbital Velocity		Mean Distance to Sun (km)	Mass (kg)
	Km/s	mph		
Mercury	47.89	$1.07x10^5$	$5.83x10^7$	$3.30x10^{23}$
Venus	35.03	$7.82x10^4$	$1.08x10^8$	$4.89x10^{24}$
Earth	29.79	$6.66x10^4$	$1.50x10^8$	$5.90x10^{24}$
Mars	24.13	$5.39x10^4$	$2.27x10^8$	$6.42x10^{23}$
Jupiter	13.06	$2.91x10^4$	$7.78x10^8$	$1.90x10^{27}$
Saturn	9.64	$2.15x10^4$	$1.43x10^9$	$5.68x10^{26}$
Uranus	6.81	$1.52x10^4$	$2.87x10^9$	$8.85x10^{25}$
Neptune	5.43	$1.21x10^4$	$4.50x10^9$	$1.02x10^{26}$

Table 1: Characteristics of our solar system [3]. Sun's mass is $1.89x10^{30}$ kg.

Moving Data into Geometer's SketchPad

The orbital data taken at equal time intervals through visual tracking within the IP window can be screen dumped into the Geometer's Sketchpad (GSP) for data analysis. This is done by a) clicking on the *PrintScreen* button and capturing the visual data while still within the IP, and b) pasting the captured data into the GSP window after closing the IP window. As seen in Fig. 1, the copied IP window is embedded within the GSP window. For some planets, orbits seem more elliptical than others. The visual data from IP shows a planet's position at equal intervals. Once within GSP, these positions can be highlighted as data points. The GSP labels these points (A, B, C,..) as shown in the picture below. To get the area of a triangle, all three corner points need to be highlighted. Figure 1 shows calculated areas for different triangles. It is clear that these areas are numerically the same within the margin of errors (2^{nd} law). Some of the variations are due to the lost area underneath a triangle's base when it does not follow the arc.

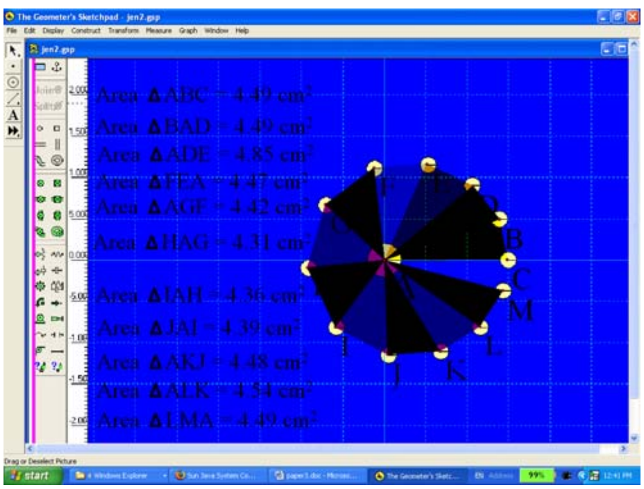


Figure 1: Computed areas for orbital tracks from IP.

Similarly, a screen dump from an IP simulation of multiple planets into the GSP allows observation of Kepler's 3^{rd} law, which states that for each planet the square of the period (T^2) is proportional to R^3; or $(T_1/T_2)^2 = (R_1/R_2)^3$ when we consider two planets. Figure 2 shows orbits of 3 planets that can be used to examine this relation. Periods (T_1, T_2, and T_3) for planets' orbits were measured as 55, 104, and 289 (days) within the IP while the semi-major axises (AB/2, AC/2, and ED/2) were calculated by the GSP. The calculations shown on the figure have been generated by the GSP for each pair of these planets. The calculations on the left $(R_1/R_2)^3$ are from GSP while those on the right $(T_1/T_2)^2$ are from IP. Ratios match verifiably.

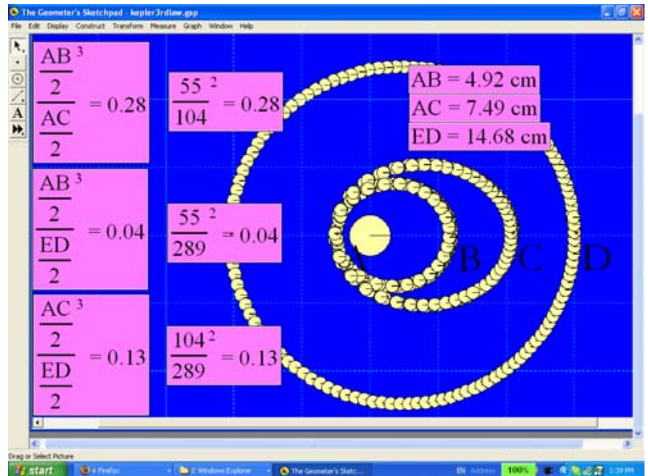


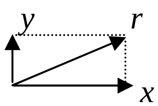
Figure 2: Examining Kepler's 3^{rd} law via GSP

Verification and Analysis by Excel

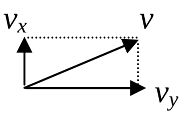
Mathematical Background: For schools with no access to IP and GSP, there is a way of repeating our experiment using Excel. Many people use Excel just to keep track of their checkbooks or to convert numerical data into graphs. We discovered that Excel could be a vehicle for us to understand details of modeling by tools such as IP. While we enjoyed running many IP simulations of our solar system, we were slowly drawn into a curiosity to understand the underlying mathematics and science. Excel gave us a chance to understand the complexity of the calculations that IP did in a matter of seconds. It taught us the hardships of manually finding the answers and helped us realize how IP technology and computational science have made complex formulas easier to handle. While IP does not require any knowledge of underlying math and science, Excel requires algebraic formulas. Yet, the needed formulas are not complex at all. For example, the entire field of simulations is based on using a simple relation: *new= old + change*. What this means is that tomorrow's temperature can be predicted as the sum of today's temperature and the amount of change expected. Another example could be: the distance to be traveled by a car in the next hour can be stated as the sum of the current distance plus the amount of change expected in the next hour.

Let's establish a notation for the rest of our mathematical demonstration. Since the data we transferred from IP into GSP involved locations (r) and velocities (v) of planets as a function of time (t), we can compute them via $r_{new} = r_{old} + dr$ and $v_{new} = v_{old} + dv$, where dr and dv represent changes in r and v. The key here is, of course, how to calculate the changes dr and dv. We need to know what governs the change in our environment and that is where science comes into the picture.

Another notation we need here is to consider both the distance (r) and the velocity (v) in terms of their two-dimensional representations as follows:



$$r^2 = x^2 + y^2$$
$$v^2 = (v_x)^2 + (v_y)^2$$



Scientific Background – Newton's Laws of Motion: While some basic scientific knowledge is needed to predict changes in our universe, the IP experimentations made this a curiosity for us, not a task. Although we had not taken any physics courses, our research revealed interesting facts that we did not know about our universe. We learned that objects attract each other via a gravitational force (F), first formulated by Isaac Newton [4] as $F=G{\cdot}M{\cdot}m/r^2$; where G is a Universal Constant and M and m are masses of objects separated by distance r. Another important discovery by Newton states that, a particle (with mass m) subject to such an external force (F) would move with acceleration (a) that is equal to F/m. What this means is that acceleration of an object under the influence of an external force (F) is inversely proportional to its mass (m): the greater the mass, the less the acceleration. Mass of an object can be considered as a measure of its resistance to external forces. The amount of force needed to accelerate an object of 1 kilogram in the amount of 1 meter per second is called 1 Newton.

Putting together these two laws by Newton, we find gravitational acceleration (a) to be equal to $G{\cdot}M/r^2$. According to physics, this would reveal all that is needed to compute changes in r and v within Excel calculations. Consulting our everyday culture about driving, we know that acceleration is a measure of change in velocity with respect to time, which can be interpreted as $a= dv/dt$. And, we also know that velocity is a measure of change in distance with respect to time as $v= dr/dt$. From these, we can see that $dr= v{\cdot}dt$ and $dv= a{\cdot}dt$. Since the planetary motion in IP was in two-dimension, we need two points (x and y) to specify planets' locations, velocities, and accelerations in each of these directions. As a result, we need to compute:

$$x_{new} = x_{old} + v_x{\cdot}dt \quad v_{xnew} = v_{xold} + a_x{\cdot}dt$$
$$y_{new} = y_{old} + v_y{\cdot}dt \quad v_{ynew} = v_{yold} + a_y{\cdot}dt.$$

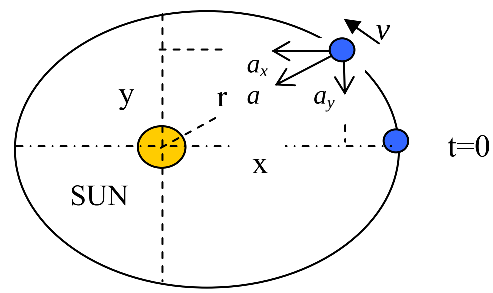


Figure 3: Relationship between directional components of r and a. Here, $a_x = (x/r) \cdot a$ and $a_y = (y/r) \cdot a$, where $a = 1/r^2 \cdot 1.26x10^{14}\ N{\cdot}km^2/kg$ [4].

Excel Computations: Table 2, below, shows a partial view of the Excel calculations of the above equations for the planet Earth. Initial values of x, y, v_x, and v_y are based on the solar data in Table 1. At t=0, we assumed that the Earth's orbital velocity is given by v_x=0 and v_y= 29.79 km/s and its position is given by y=0 and x= 1.50x10^8 km, as shown in Fig 3.

t (days)	**v_x (km/hr)**	**x (km)**	**v_y (km/hr)**	**y (km)**
0	0.00E+00	1.50E+08	107000	0
5	-8.71E+03	1.49E+08	1.07E+05	1.28E+07
10	-1.74E+04	1.47E+08	1.06E+05	2.56E+07
15	-2.61E+04	1.44E+08	1.05E+05	3.82E+07
20	-3.47E+04	1.40E+08	1.02E+05	5.05E+07
25	-4.31E+04	1.34E+08	9.94E+04	6.24E+07
30	-5.12E+04	1.28E+08	9.57E+04	7.39E+07
35	-5.89E+04	1.21E+08	9.12E+04	8.48E+07
40	-6.62E+04	1.13E+08	8.61E+04	9.51E+07

45	-7.31E+04	1.04E+08	8.03E+04	1.05E+08
50	-7.94E+04	9.49E+07	7.40E+04	1.14E+08
55	-8.52E+04	8.47E+07	6.71E+04	1.22E+08
60	-9.02E+04	7.39E+07	5.98E+04	1.29E+08

Table 2: Earth's orbital motion with Excel. See Fig. 4 for complete set of data.

Figure 4 shows Earth's orbital track based on the computations in Table 2. It may not be obvious that the orbit is an ellipse. It is clear, however, that it is not a circle. Both radii in *x* and *y* directions are different and the center of the track is not where the Sun is. Additional mathematical work can be done to show the track to be elliptical; however, we have seen visually, through IP and Excel, the imperfect shapes of the orbits as predicted by Kepler long time ago. At the time of Kepler, deviations from perfect (divine order) met with much resistance that he had to go a great length to prove otherwise.

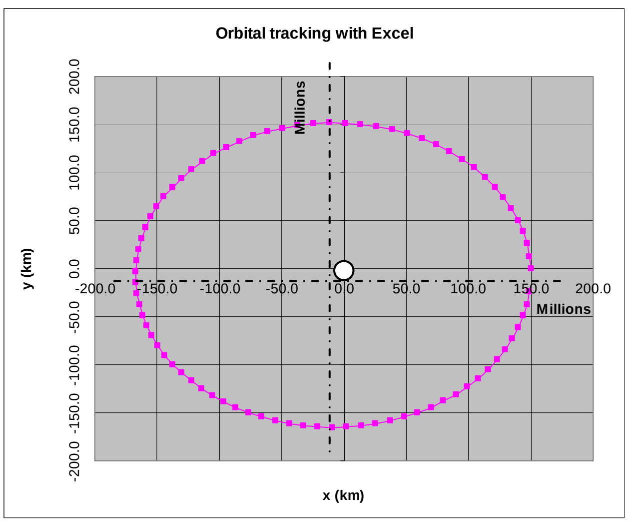


Figure 4: Orbital tracking for Table 2 (0<t<400 days).

It is important to note that Excel calculations present some variations when the time step (*dt*) is changed. What is shown in Fig.4 may not be the most accurate track, but qualitatively it is representative of a planet's orbit. Some planets have more elliptically looking orbits. The above calculations are given for $dt = 5$ days, however we believe smaller time steps (*i.e.,* $dt=1$ day) would produce more accurate tracks. Although we initially took IP and Excel computations for granted, the sensitivity of results to the time step sparked awareness in us about the accuracy of mathematical and computational results. This topic of computational accuracy is further examined in another paper by the authors.

Conclusion

Through use of mathematical and computational tools such as IP, GSP, and Excel, we have observed Kepler's laws. Ability to conduct controlled experiments on a system far from our sight and touch gave us tremendous excitement to do scientific investigation. We learned some basic laws of physics without taking a course in physics. Our curiosity drove us to better understand details of mathematical and computer modeling embedded in simulation tools such as IP. We discovered that Excel is more than a spreadsheet program. We hope that our demonstrations here can be used in mathematics and science courses as an example of how math and science can be presented in the same context.

While Kepler's work was an important historical step leading to Newton's discovery of the gravitational law, Newton's laws in return help all of us understand and verify Kepler's laws. When using tools such as IP, students need not to know the Newton's laws. This enables students to study behavior of planets without the deep knowledge of physics, or mathematics.

Computational modeling offers a rare chance to integrate mathematics and science education. With necessary accuracy and validation, computational modeling is also used in scientific and industrial research, especially when dealing with systems that are too complex to study analytically, too expansive to observe, and too dangerous to experiment. We agree with a recent Presidential report [5] states that CMST is one of the most important fields of the 21st Century.

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