

LAYERS OF ABSTRACTION: THEORY AND DESIGN FOR THE INSTRUCTION OF LIMIT CONCEPTS

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Imagine asking a first-semester calculus student to explain the definition of the derivative using the epsilon-delta definition of a limit. Given the difficulty of each of these concepts for students in such a course, you might not be surprised at the array of confused responses generated by a question requiring understanding of both. Since the central ideas in calculus are defined in terms of limits, research on students' understanding of limits and the ways in which they can develop more powerful ways of reasoning about them has significant implications for instructional design. Throughout this paper we will focus on calculus courses intended as an appropriate introduction for students who have never seen limits or derivatives and that is not intended to be a rigorous treatment of analysis. The following typical response to the question relating the definitions of limit and the derivative illustrates the confusion that students exhibit when trying to make such connections. This response was offered by an A-student, who we will call Bob, during a clinical interview late in a first-semester course:

Your epsilon - this - the slope of this tangent line. You want to pick a set of x 's, and that's here [*points at graph*]. This x , it's barely changing such that it's equal to or less than this tangent line. That would be your delta. The slope - oh, OK. The slope of this tangent line [*points at tangent*] - that's epsilon. The slope of this line [*points at secant*] that you're making is your delta at 2. Take a delta - a slope of this line [*points at secant*] less - such that it is less than the slope of this tangent line.

Bob's language is confused, but it seems he was identifying epsilon as the slope of the tangent line and possibly both x and delta as the slope of a secant line and indicates that he wants the latter to be smaller. It is quite likely that this question was beyond Bob's conceptual resources and that he was simply trying to make any connection possible to appease the interviewer. Had he been present, Bob's professor might have been rather discouraged by this response given the efforts he made in class, during special study sessions, and through creatively designed homework to help his students understand the formal definitions of both limits and derivatives.

Only a few seconds later, however, the interviewer asked Bob to explain the same idea in terms of approximation and received this response:

There will be - there could be a difference in the slopes of these lines. You could say that the slope of this line [*points at secant*] is approximately equal to this [*points at tangent*] with a margin of error of such and such, and that margin of error can be less than that [*points at the word "bound"*]. You can choose a slope that's less than the margin of error - less than whatever you need it to be.

When asked to explain his use of language about "error" in more detail, Bob explained

Your margin of error is here [*holds up hands facing each other to indicate a distance*] and here's your limit [*waves one hand*] and you have to be at least in so far closer to it [*waves other hand across the space in between*]. You can always get closer to it, you know? That's the way I was looking at bounding. You can always get closer to it.

Bob's characterization of the limit is noticeably different in this excerpt. He described approximating the slope of a tangent line using the slopes of secant lines with an error that can be made smaller than some predetermined bound. Much of the logic involved in this statement is identical to the logic of the epsilon-delta statement that he completely failed to interpret only moments earlier.

What did Bob understand about limits? What about derivatives? What bearing did his understanding of limits have on his understanding of derivative? In this chapter, we will explore the pedagogical implications of this sort of discrepancy in students' ability to articulate mathematical concepts involving limits in a wide variety of situations. This will lead into a design perspective on how we might better help students learn and use limit concepts. Before engaging this task, however, we will address how abstract concepts develop in general and identify various goals for teaching limits.

The Nature and Process of Abstraction

One of Jean Piaget's most forcefully repeated conclusions from careful observations of the nature of abstraction relates to the source of abstract concepts. Specifically, he argues that the source of conceptual structure such as that found in mathematics is an individual's *actions* or *coordinations of actions* on physical or mental objects (Piaget, 1970a, 1970b, 1975, 1980, 1985, 1997). As illustrated in Figure 1, the significance of this statement is that it emphasizes actions rather than other potential sources, such as objects, their properties, or even relationships among objects. To serve as the source of an abstracted concept, such actions must be engaged repeatedly while receiving and incorporating feedback under the specific constraints of a system that is being explored. Piaget (1970b) used the structure of the algebraic group as a prototype for all conceptual structure to emphasize the way in which actions embody structure that can be abstracted by the individual. He emphasized the role of operations (such as physically or mentally performing a symmetry transformation of a figure) and ways in which they can be coordinated (such as realizing an associative property or inverse condition) in addition to the elements with their properties and relations. Conceptual structure, such as that representing a dihedral group, is formed as a whole from the inseparable interplay between these elements and operations.

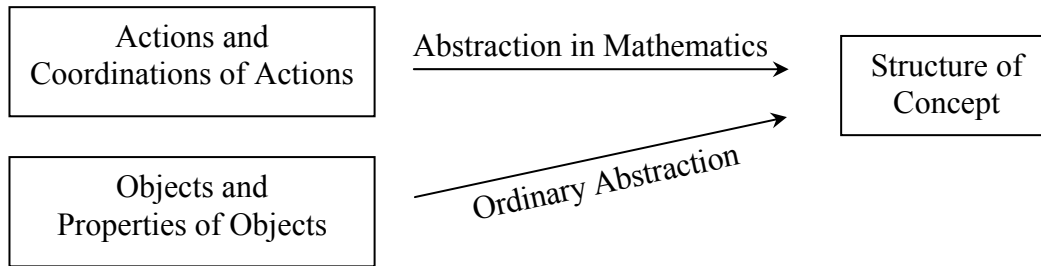


Figure 1. The source of complex abstract concepts are actions and coordinations of actions.

Piaget's characterization of the process of abstraction applied to limit concepts suggests three important features of instructional activity. First, the structure of understanding we hope our students will achieve should be systematically reflected in the actions we ask them to perform. Since these activities form the basis of conceptual understanding and thus also precede such understanding, they must be stated in terms more accessible to students than formal definitions. If a formal understanding (such as an ability to interpret and apply epsilon-delta definitions in a variety of contexts) is to eventually develop, it will be built on conceptual structures that already make sense to students because of their previously internalized activity. On the other hand, if students will never be expected to develop such formal understandings, then their conceptual structures and abilities can still reflect rigorous and appropriate mathematics. For example an engineering student, such as Bob who was introduced at the beginning of this chapter, might be able to develop a general understanding of techniques by which the error for an approximation may be made smaller than some required bound without this being formalized as finding a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

The second feature of instructional activity we infer from Piaget's theory of abstraction is that students' actions should be repeated and coordinated in ways that help them attend to feedback obtained from the inherent constraints of the system being explored. In the preceding example, the dependence of delta upon epsilon is a key feature of the structure. Students struggle with this dependence, however, since it moves in the opposite direction from the action of the function (i.e., it moves from a condition in the range to a condition in the domain). A student such as Bob may only attend to the appropriate dependence due to encountering a difficulty that otherwise arises, for example, through a real need to find an approximation with sufficient accuracy for some purpose.

The third implication we draw from Piaget's theory is that instruction on the limit concept should not be isolated, but extend throughout its many applications in the calculus curriculum in ways that foster mutual growth. The concepts defined in terms of limits provide fertile ground for continued exploration into important issues related to limits. Reciprocally, an emerging understanding of the depth of limit structures can help guide students' explorations into these other concepts.

Goals for Teaching Limits

Instructional decisions regarding the teaching of limits will ideally follow from specific objectives for students' learning. We will briefly outline a small number of possible goals and trace research related to the implications of pursuing each one.

Exposure to formal definitions and proofs. A possible objective for the instruction of limits is for students to develop facility with formal epsilon-delta and epsilon- N definitions and arguments. In fact, the *Principles and Standards*, published by National Council of Teachers of Mathematics (NCTM) argue that throughout their mathematical careers, students should continually engage in proof and argumentation. Construction of simple epsilon-delta proofs provide opportunities to interpret and then use definitions in a rigorous fashion (cf., Edwards and Ward, this volume, for a discussion of the role of definitions in formal proofs). Other objectives that would lead to a similar approach include fostering students' understanding of limit concepts in terms of epsilon-delta (or epsilon- N) arguments preparing students for more advanced mathematical study and establishing a rigorous foundation for the entire calculus curriculum. Such goals were especially pursued during the 1960's and early 1970's in efforts to increase the rigor of mathematics curricula to support a growing demand for scientists, engineers, and mathematicians.

Most calculus textbooks include a section on the formal definitions of limits, providing basic epsilon-delta and epsilon- N definitions, some pictures and intuitive explanations using graphs of functions and sequences, simple existence proofs for specific limits, and proofs of basic properties of limits (such as linearity). These ideas are typically presented in the text shortly after limits are introduced, but since most introductory calculus courses are not intended to provide a rigorous treatment of analysis, they are rarely raised afterwards. Several education researchers and curriculum committees have concluded that carrying formal limit proofs forward throughout an introductory calculus sequence might be successful in preparing a small number of the most talented students for further studies in advanced mathematics, but it leaves the vast majority of students with little more than a procedural understanding and an impression of mathematics as personally incomprehensible (Davis, 1986; Tall, 1992; Tucker & Leitzel, 1995). It is unclear whether introducing formal definitions even conveys to students a sense that there is a rigorous foundation for the mathematics. Consequently, these definitions and proofs are often de-emphasized in current curricula and courses as explicitly recommended in the report of the content workshop for the MAA publication *Toward a Lean and Lively Calculus* (Tucker, 1986).

A more modest goal for introducing limit proofs than providing a rigorous treatment of the entire calculus curriculum is to engage students in a limited amount of formal mathematical argumentation. Unfortunately, many instructors find little time to devote to this goal under the pressures of an expansive curriculum. As a result, most students only learn the basic patterns to complete simple algebraic proofs or learn the rules and peculiarities of a particular representation (e.g., games where you "keep the graph in the box" on a calculator or player 1 challenges with an ϵ and player 2 finds a δ) without understanding the connections to other representations, potential applications, or other content in the course (Jacobs, Larsen, & Oehrtman, 2003).

From Piaget's characterization of the process of abstraction, we can understand some of the difficulties students have with formal limit concepts. Instruction that begins with formal definitions attempts to move in the opposite direction from which abstraction

naturally occurs. When students are first exposed to the concepts in calculus, there is no conceptual structure through which they can meaningfully interpret key features of formal limit structures. Based on Piaget's theory of abstraction and refined through a series of clinical interviews with students, Cottrill et al. (1996) have proposed a progression of actions that students must abstract, generalize, and relate to one another in order to construct such a conceptual structure. For the limit $\lim_{x \rightarrow a} f(x) = L$, they suggest

that students must first abstract the actions of evaluating f at points near a , then develop and coordinate domain and range processes of x approaching a and $f(x)$ approaching L . Then this coordinated structure must be reinforced by performing actions on limits, such as by considering limits of combinations of functions. Only at this stage in Cottrill et al.'s framework are students able to reconstruct these coordinated processes in terms of inequalities, apply a consistent understanding of the universal and existential quantifiers, and develop a complete epsilon-delta conception for a specific situation.

Attempts to support students' understanding of a formal definition with an intuitive rephrasing such as "You can make $f(x)$ arbitrarily close to L by making x sufficiently close to a " also provide neither appropriately structured activity nor underlying meaning, so are likely to fail as well. Instead, under the burden of making some sense out of what is being said, students attach simpler meanings to these phrases. In interviews with students throughout three semesters of calculus being exposed to such language, nearly all interpreted the modifiers "arbitrarily" and "sufficiently" in the simplest way possible: as indicators of degree (Oehrtman, 2002). To them, "sufficiently small" meant "very small" and "arbitrarily small" meant "very very small." These students did not have any experiences from which the intended logical entailments of these phrases could be generated.

If students are not expected to use epsilon-delta definitions and arguments throughout a course, the corresponding conceptual structure is neither continually reinforced nor developed for use as a powerful tool. Consequently, it is unclear how formal limit definitions and proofs could guide students' exploration into subsequent topics without offering a nearly complete analysis course.

Intuitive understanding. Most secondary and introductory undergraduate calculus courses and textbooks take an approach to limits that focuses on intuitive ideas and phrasings, such as "when x gets close to a , $f(x)$ gets close to L ." Even if formal definitions are introduced and used to prove some basic properties of limits, they are de-emphasized or abandoned when advancing to subsequent concepts, even those that are defined in terms of limits. The definition of derivative is rarely treated in terms of epsilons and deltas in an introductory calculus course, for example. One purpose of treating limits with an intuitive approach is to provide a common and accessible introduction to other concepts throughout the course (cf., Speiser & Walter, this volume). Derivatives are discussed in terms of secant lines where the points are made closer and closer to each other, and definite integrals are defined as summing products over intervals that get smaller but increase in number. Another objective leading to similar approaches is the need to teach a number of techniques for algebraic computations that will be used later in the course, such as finding certain derivative formulas, determining specific values for improper integrals, or applying convergence tests for infinite series. Since these skills do not require students to understand epsilon-delta and epsilon- N structures,

formal definitions are often de-emphasized and intuitive descriptions of limits viewed as sufficient.

When subsequent topics are introduced through an informal understanding of limits, the role of limits is typically suppressed. Operationally, the limit concept is often concealed both conceptually by definitions of derivatives in terms of slope, definite integrals in terms of area, Taylor series as actual sums, etc. Corresponding to this conceptual shift, limits may also be de-emphasized notationally. For example, $f'(a)$ may be described as the slope of the tangent to the graph of f at $(a, f(a))$, the definite integral

$\int_a^b f(x) dx$ may be described as the area under the graph of f on the interval $[a, b]$, or the

Taylor series $\sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x-a)^n$ as being equal to $f(x)$ for most functions students will

see. Limit notation is absent from all of these descriptions and limit structures (as encapsulated by epsilon-delta or epsilon- N definitions) are even further in the background. Intuitive images such as tangent lines, areas, and infinite sums are often used as a proxy for limits since they are conceptually accessible to students and can be extremely powerful for intuitive reasoning (Monk, 1987, 1992; Rodi, 1986; Tall, 1992; Thompson, 1994).

Informal language and reasoning about limits can also lead to misconceptions for even advanced students who are supposedly equipped with the formal tools to avoid such errors. Twenty-two students in their final year of university mathematics and who had dealt with the formal epsilon-delta definition of limits for two years were asked the following question:

True or false: Suppose as $x \rightarrow a$ then $f(x) \rightarrow b$, and as $y \rightarrow b$ then $g(y) \rightarrow c$. Then it follows that as $x \rightarrow a$ then $g(f(x)) \rightarrow c$.

All but 1 of these 22 students responded “true” and refused to change their answer even when pressed (Tall & Vinner, 1981). Whether considered explicitly or subconsciously, the logic of this statement as typically verbalized establishes a false syllogism by stating “If x approaches a then $f(x)$ approaches b , and if y approaches b then $g(y)$ approaches c .” If the first premise holds, then $f(x)$ satisfies the hypothesis of the second premise, i.e., it qualifies as a y that approaches b , thus leading to the incorrect conclusion that $g(f(x))$ approaches c . Additionally, the arrows and their verbalizations as “approaches” or “goes to” are represented in the same way for both dependent and independent variables in this problem. The suppression of different technical meanings by using the same notation for both contributes to students’ misconceptions.

In another study on the use of intuitive dynamic language, when faced with challenging problems involving limiting situations, students did not rely on images of motion to reason about the problems (Oehrtman, 2003). This is particularly surprising given the predominance of motion language used when talking about limits and abundant proclamations that intuitive, dynamic views of functions should help students understand limits. While students frequently used words such as “approaching” or “tends to,” these utterances were not accompanied by any description of something actually moving. When asked specifically about their use of such phrases, students denied thinking of motion and gave an alternate explanation for their words.

In terms of abstraction, the informal approaches to limits described in this section are susceptible to a similar problem as purely formal approaches. Without other supports, they do not provide students with a structure that can guide their investigation of the relevant mathematics of subsequent topics. Instead of providing an incomprehensible structure, they provide little to no structure, but the result is still that students are left to construct an understanding based on disjoint and possibly unguided connections and images. The work of understanding subsequent topics is then shifted to representations specific to each one (a lack of gaps for continuity, steepness for derivatives, area for definite integrals, etc.). Each of these understandings, then are bound to a specific representation (typically graphical), and for students to reason conceptually requires a translation back and forth between that representation and the problem context. It is difficult for students to see and work with the commonalities between these images as required, for example, to understand the fundamental theorem of calculus. Further, since the central concepts in calculus are defined in terms of limits, important aspects of these structures are also lost to the same degree that limits are de-emphasized.

De-emphasis of limits and alternative foundations of calculus. Due to the well-documented difficulty of limit concepts, several researchers and reformers have suggested providing a more intuitive starting point for calculus. The most common is based on using infinitesimals, rounding, and local linearity. This approach mirrors some aspects of Newton's reasoning with fluxions and fluents and Leibniz's notational encapsulation of infinitesimal quantities. Concerns about lack of rigor are addressed by referring to Robinson's set-theoretic work in the 1960's establishing nonstandard analysis as a logical foundation for an infinitesimal approach. Promising aspects of infinitesimal instruction are that the foundational concepts are accessible and that, in some cases, students used those ideas as integral parts of their reasoning.

Citing historical and cultural difficulties related to concepts of function, limit, infinity, and proof, Tall (1986, 1990, 1992) suggests that a better cognitive starting point for calculus might be "local straightness." Students are introduced to tangents via magnification of the graph of a function at a point. This approach, he suggests, allows for the investigation of a rich source of concepts: different left and right gradients, functions that are locally straight nowhere, etc. Students taught with this approach were much better at recognizing, drawing, and reasoning about graphical information for derivatives than students in a control group. On the other hand, they tended to describe a tangent as passing through two or more very close points on the graph. At least part of these students' difficulties seems to be conflation of the tangent line and the actual graph caused by the appearance of the graph as a straight line after sufficient magnification.

In the infinitesimal approach, computations are performed using an infinitesimal element, ε , and standard algebra extended to the infinitesimals. This process is followed by "rounding off" infinitesimal terms, so that an expression like $2x + \varepsilon$ is replaced by $2x$. Tangents are then treated by magnification as described above with the addition that the graph is magnified to an infinitesimal scale. Frid (1994) found that although students who were given instruction with infinitesimals did not perform significantly better on standard computations, they did use the language and notation of rounding as an integral part of their explanations. Whether the students' use of everyday language was of help or a hindrance depended on the extent to which they integrated that informal language with technical language or symbols in ways congruent with the corresponding concepts.

Michèle Artigue (1991) conducted a study with 85 third-year university students enrolled in multivariable calculus and physics courses to investigate their understanding and use of differential elements. In their courses, students were provided with a tangent linear approximation definition, which dominated their descriptions of differentials. At the procedural level, however, they reverted to treating differentials algebraically in algorithms involving partial derivatives and Jacobian matrices. Students were not able to identify conditions in specific contexts necessitating the use of differentials and gave incorrect justifications about convergence of approximations based on convergence of geometric “slices” in a diagram. Such arguments also may be common for students who have not received infinitesimal instruction, however. Thompson (1994) observed advanced mathematics students incorrectly justify the fundamental theorem of calculus by arguing geometrically that the shape of a three-dimensional object with thickness Δx converges to a two-dimensional object as $\Delta x \rightarrow 0$. Oehrtman (2002, 2003) classified a ubiquitous category of such reasoning in terms of “collapsing dimensions” among freshmen calculus students and secondary mathematics teachers in a wide variety of problem contexts.

Ideas of local linearity contain conceptual pitfalls when used to supplement a standard treatment of calculus. In a class regularly exposed to descriptions of zooming in on graphs, students did not ever develop these concepts for use in any of their own explanations about limits (Oehrtman, 2002). When directly probed about what they would see when zooming in on the graph of a function, only 10 out of 77 gave a response that was relevant to the mathematics, and these were all incorrect, suggesting that one would see a horizontal line because the vertical change is reduced to a very small amount (although this argument seems to imply that one would see a horizontal line). All of the other students attended to non-mathematical interpretations such as images of the line becoming thicker or blurrier under magnification or that you would see individual calculator pixels or atoms of paper. This indicates that images of zooming did not provide these students with sufficient structure to guide their reasoning and the related instructional process lacked the necessary feedback to prevent major misconceptions. Additionally, subsequent analysis-based mathematics courses are rarely taught in terms of nonstandard analysis and science and engineering rarely use mathematical models that incorporate infinitesimals.

A Design Approach to Limit Instruction

The main thrust of this chapter is to frame a set of objectives related to the learning of limits taking into account many of the goals introduced above and to outline an instructional approach based on research and refined through several teaching experiment cycles. One of our main objectives is to base the instruction on activities that are conceptually accessible to students. As discussed above, this has been achieved by others, notably through infinitesimal approaches, and we have drawn from their successes. A second goal is to structure students’ understanding in ways that reflect formal definitions. The purpose of this is to lay conceptual groundwork from which formal understandings may later emerge but not necessarily to provide those formalizations themselves. Such an approach could, of course, leave open the option for an instructor to develop these definitions at an appropriate time. Third, we strive to establish an instructional approach for limits that serves as a guide for the investigation of all other concepts defined in terms

of limits in ways that enhance exploration of their underlying structures. Finally, the approach should allow and encourage flexible application in all representations (algebraic, graphical, numerical, contextual/descriptive, etc.). The diversity of these goals leads to the consideration of an important additional constraint: we require an approach that is coherent. That is, the treatment should be mutually reinforcing across the entire calculus curriculum, and the process of achieving each goal outlined above should support the attainment of the others.

A design process. From a design perspective we have sought to achieve these goals via the following process:

1. Identify the mathematical structures (elements, operations, relations that result from coordinating operations, etc.) that must be reflected in the instructional activities.
2. Identify a structurally equivalent conceptual system and language base that is accessible to students. This is achieved by documenting students' natural reasoning, developing possible frameworks of mathematical expressions for this reasoning, then evaluating the effectiveness of structuring students' activities around these mathematical versions of their natural reasoning.
3. Develop, test, and refine instructional activities in which students apply the framework to particular applications. Students work in groups on structurally similar problems in a variety of contexts and then present results to each other, reinforcing the structure across novel contexts and problems. Design whole-class discussion to elicit the common features across all applications. Initial activities should focus on familiarizing students with the language, notation, and procedures of the framework and assisting them in choosing and applying its tools (e.g., focusing on types of questions generally asked, common procedures that may be used, and relevant representations of the results). Later activities should encourage students to reason through solving problems on their own.
4. Repeat Step 3 for a variety of applications of the concept. This establishes a second level of activities in which students are encouraged to see similarities across different uses of the concept and develop a more general and robust abstraction of the concepts.
5. Design tasks to foster formalization as an end result. This includes naming or symbolizing a structure that has already been abstracted and can lead to discussion and use of formal definitions and proofs.

The overarching principle is that students should engage in multiple activities that reveal and encourage the abstraction of a common structure, and the results of many such abstractions should share common features to allow for further levels of abstraction. At each level, students should participate in experientially real activities designed to engage them in the relevant structures of the underlying mathematics (although not necessarily the formal representations) and in seeing common structures across multiple experiences. This allows an abstract understanding to emerge over a long period of time with significant reinforcement at a variety of conceptual levels. The concept may be formalized near the end of this process as a way to concisely capture a well-understood structure.

In the case of limits, a particular application developed in an iteration of Step 3 would stem from the limit structure involved in the definition of the derivative (see Figure 2). The activities are designed to reflect the predetermined limit structures to provide an

appropriate source for later abstraction. Different groups of students present their work detailing the operations and relations involved in applying the limit framework to different rate of change contexts. Although the contexts are different, the underlying structure is the same, and classroom discussion is focused on drawing out the common features. As this is repeated for the limit of a function at a point, the definite integral, the fundamental theorem of calculus, Taylor series, etc., there is variation in the structures of these different topics but certain consistencies in the underlying limit structures.

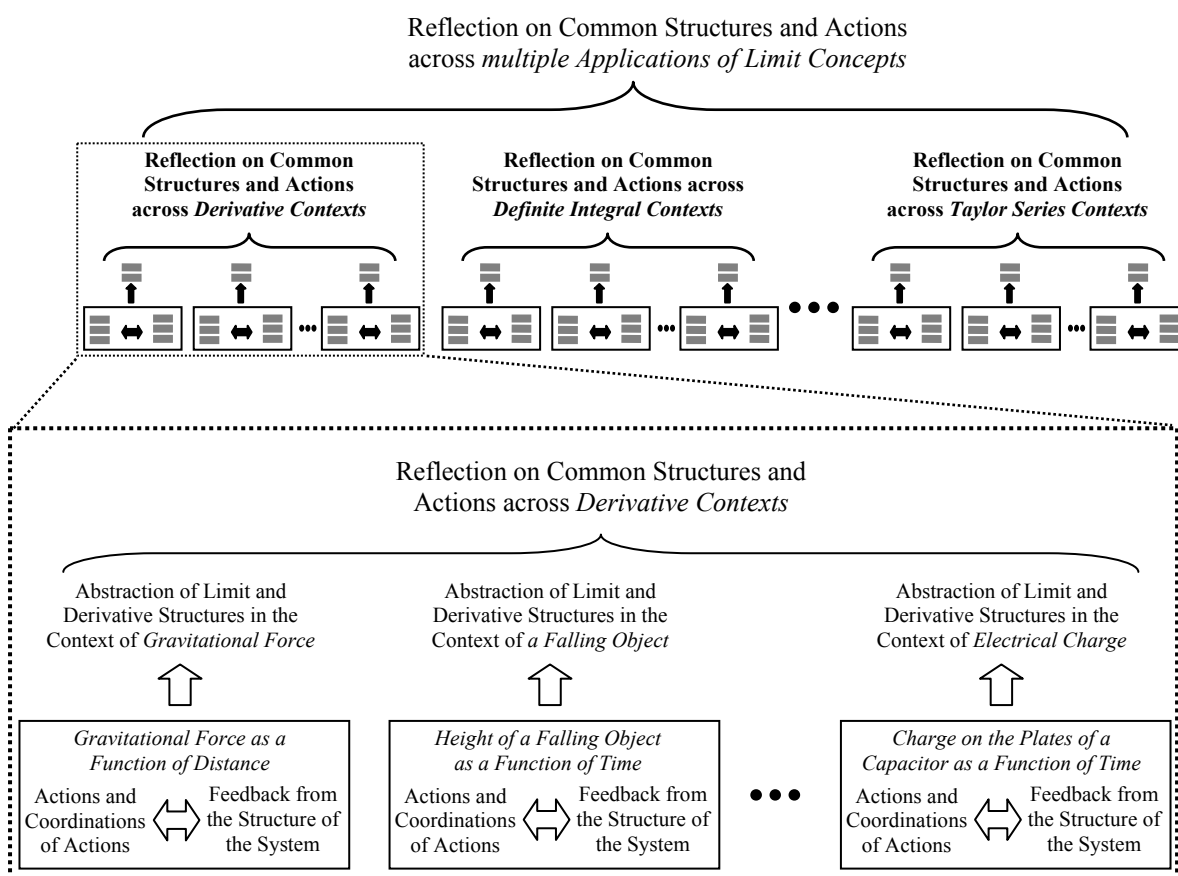


Figure 2. Layers of Abstraction: A common structure for limit concepts is repeated within each application then across multiple applications to provide coherence throughout the calculus curriculum.

We have identified in our goals, the formal limit definitions as capturing the underlying structure to be modeled through instruction. This means students should develop to use conceptual tools corresponding to the algebraic entities and expressions in the definitions, guided by the types of operations possible within the underlying logical connections between these expressions. We have also identified the need to apply this structure to solve problems in the contexts of other concepts within the calculus curriculum.

Students' spontaneous reasoning with approximation concepts. Formal limit definitions and structures are often considered beyond the reach of most introductory calculus students. Students, however, often naturally reason about limit concepts in terms

of approximations in ways that are structurally equivalent to aspects of formal epsilon-delta and epsilon- N definitions (Oehrtman, 2004). For example, students may be able to construct an idea such as “the slope of the tangent line is approximated by slopes of secant lines, and the errors (differences between the two slopes) can be made smaller than any predetermined bound” even though merely interpreting an abstract statement such as “for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $0 < |x - x_0| < \delta$ then $\left| \frac{f(x) - f(x_0)}{x - x_0} - m \right| < \varepsilon$ ” may be entirely beyond their reach. Furthermore, instruction can foster the development and application of appropriate versions of such reasoning so that it may become a basis for understanding the formal statements and incorporating aspects into their reasoning with approximations (Oehrtman, 2004). For this reason we take the stance that epsilon-delta and epsilon- N definitions should only be introduced after multiple rounds of instruction that reinforce the conceptual structure of limits in different settings as depicted in Figure 2.

In a study to characterize calculus students’ spontaneous reasoning patterns while working with limits, Oehrtman (2002) collected responses to short writing assignments from an entire class of 120 students and more in-depth descriptions of students reasoning from 25-35 students from regular online writing assignments. Nine students participated in initial clinical interviews during which the interviewer prompted for detailed explanations of their reasoning about the meaning of limits through standard problems, and follow-up interviews were conducted with an additional 11 students. Approximation ideas emerged as the strongest and most frequently applied metaphor for limits in this study, and students’ reasoning while thinking about approximations were more likely to reflect the correct mathematical structures than any of the other contexts that emerged. These results may not be surprising since much of calculus is historically motivated by needs for numerical estimation techniques, and these ideas continue to influence our classroom and textbook presentations. Consider the following quote from a typical second-semester calculus student as she explains her understanding of the equality $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$. Attend to her use of the words “approximation”, “error”, and “accuracy” and how their usage matches the structures in epsilon- N convergence arguments.

When calculating a Taylor polynomial, the accuracy of the approximation becomes greater with each successive term. This can be illustrated by graphing a function such as $\sin(x)$ and its various polynomial approximations. If one such polynomial with a finite number of terms is centered around some origin, the difference in y -values between the points along the polynomial and the points along the original curve ($\sin x$) will be greater the further the x -values are from the origin. If more terms are added to the polynomial, it will hug the curves of the sin function more closely, and this error will decrease. As one continues to add more and more terms, the polynomial becomes a very good approximation of the curve. Locally, at the origin, it will be very difficult to tell the difference between $\sin(x)$ and its polynomial approximation. If you were to travel out away from the origin, however, you would find that the polynomial becomes more and more loosely fitted around the curve, until at some point it goes off in it's own direction and you would have to deal once again with a substantial error the further you went in that direction. Adding more terms to

the polynomial in this case increases the distance that you have to travel before it veers away from the approximated function, and decreases the error at any one x -value. Eventually, if an infinite number of terms could be calculated, the error would decrease to zero, the distance you would have to travel to see the polynomial veer away would become infinite, and the two functions would become equal. This is a very important and useful characteristic, as it allows for the approximation of complicated functions. By using polynomials with an appropriate number of terms, one can find approximations with reasonable accuracy.

This student received no special instruction related to ideas about approximation, yet the language of approximation, errors, and accuracy figured prominently and systematically in her reasoning. Furthermore, the structure of these ideas for this student reflect a sophisticated understanding of limits and is integrated with her understanding of various aspects of Taylor series, such as the relationship between graphs of a function and several Taylor polynomials, pointwise convergence, and the radius of convergence. These types of statements were common among students trying to make sense out of limit concepts in their own language (Oehrtman, 2002).

The main components of students' spontaneous use of approximation ideas to reason about limits consisted of an unknown actual quantity and approximations that are believed to be close in value to the unknown quantity. For each approximation, there is an associated error,

$$\text{error} = | \text{unknown quantity} - \text{approximation} |.$$

Consequently, a bound on the error allows one to use an approximation to restrict the range of possibilities for the actual value as in the inequality

$$\text{approximation} - \text{bound} < \text{unknown quantity} < \text{approximation} + \text{bound}.$$

An approximation is contextually judged to be accurate if the error is small, and a good approximation method allows one to improve the accuracy of the approximation so that the error is as small as desired. An approximation method is precise if there is not a significant difference among the approximations after a certain point of improving accuracy.

The structure of this schema parallels the logic of epsilon- N and epsilon-delta definitions of limits. For the latter, bounding the error corresponds to the statement “then $|f(x) - L| < \varepsilon$ ”. The need to obtain *any* predetermined degree of accuracy evokes the requirement that the condition hold “for any $\varepsilon > 0$.” A mechanism to generate better approximations corresponds to the phrase “there exists a δ such that whenever $0 < |x - a| < \delta$.” Linking these structures together gives the practical statement of being able to find a suitable approximation for any degree of accuracy on the one hand and the formal epsilon-delta definition on the other. Students' intuitive descriptions of precision such as “There will not be a significant difference among the approximations after a certain point.” reflect the structure of Cauchy convergence, if $n > N$ then $|a_m - a_n| < \varepsilon$. These structures are consistent even with a generalized definition of definite integral as a net, with partitions partially ordered by refinement. In terms of approximations such a description may be even more intuitively accessible than a simple limit definition and its restrictions on the types of approximations considered.

Instructional Activities. The types of tasks discussed in this section have been tested and refined in use with three different student populations: a standard introductory

calculus class, a supplemental calculus workshop, and a summer professional development workshop for high school teachers. Research teams videotaped all instructional settings and analyzed data to discern which aspects of the activities reinforced the desired conceptual structures and provided students with powerful reasoning tools and where potential difficulties might arise. In each round, the instructional activities were refined accordingly.

An initial task that must be accomplished by the instructional activities is to systematize students' spontaneous understandings related to approximations so that a relevant and standard set of ideas and language can be developed by the class to use in further explorations. Williams (1991) found students' exhibited strongly held sets of beliefs typically surrounding the contexts in which they were first exposed to limits and that their viewpoints were extremely resistant to change, even in response to explicit discussions about contradictory examples. Students viewed counterexamples as minor exceptions rather than reasons to abandon an incomplete concept and evaluated the appropriateness of any particular conceptualization based on its usefulness in a given setting rather than on its rigor, consistency, or correctness. These are hallmarks of spontaneous reasoning which is not volitional or structured (Vygotsky, 1987). The development of students' scientific concepts alongside their spontaneous concepts can be slow and difficult, but in any instructional process related to limits, something similar will be necessary. The key is to have a strong set of spontaneous concepts (as is the case with students' approximation ideas) to enable and mediate this process. To accomplish this, we have developed a variety of heavily "scaffolded" tasks (tasks with significant initial instructional support designed to be gradually removed throughout subsequent activities as students develop proficiency). Examples are shown in Figures 3 and 4.

In the following problem, you will approximate the slope of the tangent line to a curve at a point. There are several important ideas about approximation that are embedded in these exercises that have a close relationship to the limit concept. You will need a graphing calculator or a graphing program on a computer.

Graph $y = 2^x$ on a calculator or computer over the interval $[-3, 3]$ and take careful note of the general shape of the curve. Now zoom in on the graph at $x = -1$. That is, change the window to show the graph over a smaller interval around $x = -1$, like $[-2, 0]$. Notice that the graph appears less curved and more like a straight line. If you keep zooming in around $x = -1$, the graph will appear more and more like a straight line. This is called the tangent line to the graph of $y = 2^x$ at $x = -1$. The details of tangent lines will be developed more fully later in this course. For now, you will approximate the slope of the tangent line.

1. Look at a region of the curve where it appears fairly straight but still has a slight, noticeable curvature, e.g., on $[-2, 0]$. Take a point on the curve to the *right* of the point at $x = -1$, and find the slope between these two points. (Make sure to keep as many decimal places in your calculation as possible since this exercise will require precision.)
2. Take a point on the curve to the *left* of the point at $x = -1$. Find the slope between these two points.
3. Are the two slopes from parts a and b both underestimates, both overestimates, or one of each? Explain how you know. (Hint: Use the fact that the graph of $y = 2^x$ is concave up, i.e., it curves upwards.)
4. Using your work from above, give a range of possible values for the slope of the tangent line. Using the center of this range as an approximation, what is a bound on the size of the error?
5. Explain why your bound is just an *upper bound* for the error and not *exactly* the error.
6. Zoom in and use points to the left and right of $x = -1$ to find an approximation of the slope of the tangent line with error less than 0.0001. Record your work for each computation you do.
7. Explain why any points between $x = -1$ and the points you used in Part f would result in an approximation with error less than 0.0001.

8. What other x -values can you use for the second point and have the error be less than 0.0001.

Be prepared to answer:

1. What unknown value were you approximating?
2. What were your approximations?
3. Describe what the error for each approximation was. Why is the exact value of the error impossible for you to determine?
4. How did you bound the error?
5. Explain a procedure for getting an approximation with error smaller than any pre-determined bound.

Figure 3. A scaffolded activity on the slope of a tangent line designed to reinforce approximation structures relevant to limit concepts. This task is used before limits or derivatives are formally introduced to lay a foundation for the conceptual structure.

The graph of $f(x) = \frac{\sqrt[3]{x+7}-2}{x-1}$ has a hole. Your task is to determine the location of this hole using the

approximation techniques you have learned.

1. Identify what unknown numerical value you will need to approximate. Give it an appropriate shorthand name.
2. Determine what you will use for approximations. Write out your answer algebraically.
3. Draw the graph using your entire whiteboard. Depict your answers to #1 and #2 on the graph with labels for each.
4. What is an algebraic representation for the error in your approximations? Add a graphical representation to your picture.
5. List three fairly decent approximations. For each one, give a bound for the error and use this to determine a range of possible values for the actual value. Add one of these values to your picture and depict both the error bound and the range of possible values. Don't forget to label everything!

Approximation	Error Bound	Range of Possible Values

6. Find an approximation with error smaller than 0.0001. Then describe *all* of the approximations that would have an error smaller than 0.0001. Add this to your picture.
7. For any pre-determined bound, can you find an approximation with error smaller than that bound? Explain in detail how you know.

Figure 4. A scaffolded activity on the limit of a function designed to reinforce approximation structures relevant to limit concepts.

These highly scaffolded activities introduce the students to limit structures using applications that are fully developed later in the course, such as the derivative or continuity. Other initial tasks engage students in similar activities using different applications of limits such as the definite integral structure presented as approximating how far a wind-up car would travel and infinite series presented as “mystery” sums. Although there are opportunities for some discussions about these other topics at this stage, best results were obtained by focusing group and class discussions on the limit structures within each application in order to reinforce the use of ideas about approximations, errors, and bounding errors. In each case, students were able to reason within the given context to determine whether specific approximations were overestimates or underestimates then to find an actual bound on the size of the error. They were then asked to reverse this process and find several approximations with errors smaller than pre-determined bounds. In students' work and presentations, continual emphasis that these two processes are the reverse of one another was necessary to help

students understand the distinction and when one way of reasoning would be required over the other.

Subsequent activities provide fewer step-by-step instructions for the students with the expectation that they will begin to remember or develop appropriate strategies to solve increasingly more sophisticated problems. Through the teaching experiments, we have determined that once students are able to complete introductory activities such as the ones above, they are ready to begin group work on less scaffolded tasks as shown in Figure 5. With some preliminary discussion about average rate of change and intuitive interpretations of instantaneous rate of change, these activities prepare students for the introduction of the definition of the derivative.

Instructions: You will approximate the instantaneous rate of change for one of the situations below by answering each of the following questions algebraically, numerically, and by representing each answer in your diagram:

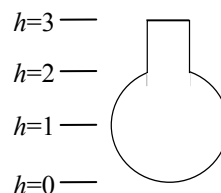
0. Draw a **large** picture of the physical situation for the value of the variable given.
1. Imagine how things are changing in this situation. List all of the quantities that you think are changing. Describe **how** they are changing.
2. On the same picture, draw several “snapshots” of the situation.
3. Label the changing and constant quantities in your drawing.
4. Describe in more detail what you have been asked to approximate.
5. What can you use for approximations?
6. What are the errors?
7. Find an approximation and a bound for the error. What is the resulting range of possible values for your instantaneous rate?
8. How can you find an approximation with error smaller than a predetermined bound?

Context 1: An object is falling according to the equation $h(t) = 100 - 16t^2$ feet (with t measured in seconds). Approximate the speed when $t = 2$ seconds.

Context 2: Approximate the instantaneous rate of change of the area of a circle with respect to its radius when the radius is 3 cm.

Context 3: The force of gravity between two objects is inversely proportional to the square of the distance separating them. Approximate the instantaneous rate of change of the gravitational force with respect to distance when two objects are 230 km apart. (Note that all of your answers will involve the constant of proportionality.)

Context 4: Approximate the rate of change of the height of water in this bottle with respect to the volume of water when the height is 1.5. (Note that your answers will involve the size of the spherical portion of the bottle.)



Context 5: The half-life of Iodine-123, used in some medical radiation treatments, is about 13.2 hours. Thus a sample that originally has $6.4 \mu\text{g}$ of Iodine-123 will decay so that the amount left after t hours will be roughly $I(t) = 6.4\left(\frac{1}{2}\right)^{t/13.2} \mu\text{g}$. Approximate the instantaneous rate at which the Iodine-123 is decaying after 5 hours.

Figure 5. Typical partially scaffolded activities developing limit and derivative structures through approximation ideas.

Eventually, students are given problems with very few prompts regarding approximation structures. Consider, for example, the problems posed in Figure 6. Typically, such contexts would be presented to students as tasks to construct a definite integral and evaluate using the fundamental theorem of calculus. The slight change in this formulation requires students to coordinate the product, sum, and limit structures of the definite integral across multiple representations. Table 1 provides brief descriptions of typical responses expected of and provided by students in previous teaching experiments for Context 2 of Figure 6.

Instructions: Draw a picture of the situation, labeling everything possible. Determine a way to approximate the quantity requested. Be prepared to explain exactly how you obtained your approximations, what your errors are, how you can bound the errors, and how you can find an approximation with an error smaller than any predetermined bound. Express your answers algebraically and numerically, labeling appropriate quantities in your diagram.

Context 1: For a constant force F to move an object a distance d requires an amount of energy equal to $E = Fd$. Hooke's Law says that the force exerted by a spring displaced by a distance x from its resting length is equal to $F = kx$, where k is a constant that depends on the particular spring. If the spring constant is $k = .155$ N/cm, approximate to within 1000 ergs the energy required to stretch the spring from a position 5 cm beyond its natural length to 10 cm beyond its natural length. (Note that $1 \text{ erg} = 10^{-5} \text{ N}\cdot\text{cm}$.)

Context 2: A uniform pressure P applied across a surface area A creates a total force of $F = PA$. The density of water is 62 lb per cubic foot, so that under water the pressure varies according to depth, d , as $P = 62d$. Approximate to within 1000 pounds the total force of the water exerted on a dam 100 feet wide and extending 50 feet under water.

Context 3: The mass M of an object with constant density d and volume v is $M = dv$. A 10-meter long, 10-cm diameter pole is constructed of varying metal composition so that its density increases at a constant rate from 3 grams per cubic centimeter at one end to 20 grams per cubic centimeter at the other. Approximate the mass of the pole with an accuracy of 100 grams.

Context 4: Fluid traveling at a velocity v across a surface area A produces a flow rate of $F = vA$. Poiseuille's law says that in a pipe of radius R , the viscosity of a fluid causes the velocity to decrease from a maximum at the center ($r = 0$) to zero at the sides ($r = R$) according to the function $v = v_{\max} (1 - r^2 / R^2)$. Find an approximation of the rate that water flows in a 1-inch diameter pipe if $v_{\max} = 2$ ft/s with an accuracy of 0.01 cubic feet per second (cfs).

Context 5: The volume V of an object with constant cross-sectional surface area, A , and height, h , is $V = Ah$. A large spherical bottle of radius 1 foot is filled to height of 16 inches. Approximate the volume of water in the bottle to within 0.01 cubic feet.

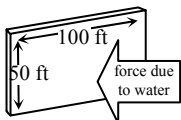
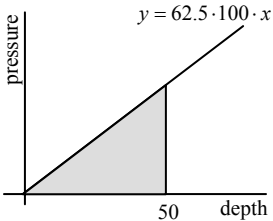
Figure 6. Typical non-scaffolded definite integral questions in terms of approximation.

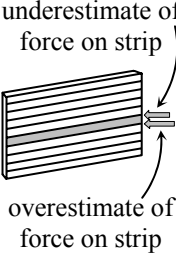
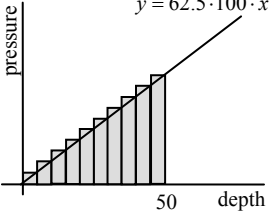
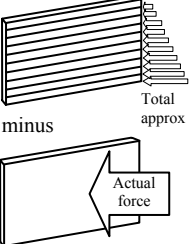
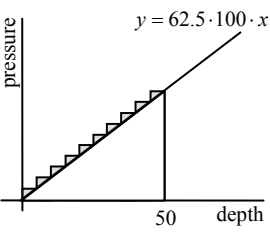
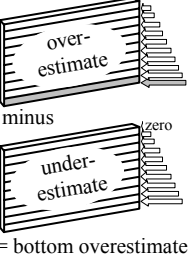
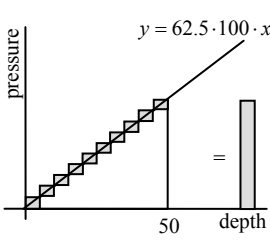
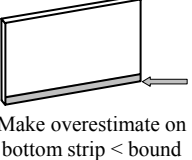
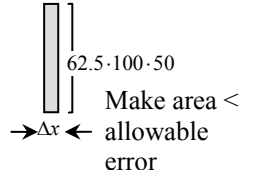
Algebraic and numerical representations of the actual value, such as shown in the shaded cells of Table 1, are usually found last by students. Even when students attempt to begin their work by computing an integral, they are typically unable to determine the correct integrand until they have found and represented several particular approximations. Other than the shaded cells, students typically proceed sequentially down the columns by finding and representing the elements of each row in the table in turn (Sealey & Oehrtman, 2005). The need to express answers using all representations arose naturally as students needed to accomplish various sub-goals. For example, to

figure out how to compute specific numerical values, students needed to carefully express their ideas on their picture of the dam, going through several revisions of their diagram and labeling. For larger computations that required a calculator or computer, students were forced to express their work algebraically in order to determine an appropriate command.

A typical group of Sealey's & Oehrtman's students wrestled with an incorrect definite integral $\int_0^{50} 62x \, dx$, then multiplied the pressure at the middle depth of the dam by its area to get what they believed was the actual answer. They then quickly determined how to find under and overestimates for partitions with ten subintervals then for 50 subintervals, and thought momentarily that they could not subdivide the partitions any further since it was now one foot each. Once they realized how to represent finer partitions algebraically and how to enter them into their calculators, they found under and overestimates for 100 and 800 subintervals (800 was the largest number of terms allowed by their calculators for computing a sum). At each of these steps, they noted that the actual force was somewhere between their values, that the error was bound by the difference, and that it was much larger than the desired 2000 pounds thus requiring further work. At this point, they proceeded to part e and eventually determined they would need 7750 subintervals. They became eager to actually try this and broke the problem into ten sub-problems with 800 or fewer terms each. All students in the group then agreed to find the sum for the first 800 terms to check their work, to work on different sums, and finally combine their results at the end. The students worked in a highly collaborative and engaged manner with these activities, and there was constant talk throughout that reflected the structure of both limits (finding approximations, determining bounds for how far off they were, and determining how to achieve the desired accuracy) and definite integrals (breaking the problem into sections where pressure is nearly constant, computing forces using products, summing the results, and developing a general Riemann sum).

Table 1. Descriptions of typical responses in multiple representations for the approximation questions applied to the question about the force of water on a dam. Students typically produced responses in the order of descending rows with the exception of the two shaded cells which were often produced last.

	Contextual	Graphical	Algebraic	Numerical
Unknown Value: The force of water against the dam	Underwater portion of dam 		$F = \int_a^b p(x) \cdot w(x) \, dx$ $= \int_0^{50} 62.5 \cdot 100 \cdot x \, dx$	$F = 7,812,500$ pounds

Approximation: Assume constant pressure across strips and use $F = P \cdot A$. Using pressure from the bottom of strips yields an overestimate. Pressure from top yields underestimate			$A = \sum_{i=1}^{10} 62.5x_i \cdot 100 \cdot \Delta x$ where $x_i = 5i$ and $\Delta x = \frac{50-0}{10} 5$	$F \approx 156,250 + 312,500 + 468,750 + 625,000 + 781,250 + 937,500 + 1,093,750 + 1,250,000 + 1,406,250 + 1,562,500 = 8,593,750 \text{ pounds}$
Error: The difference between the actual force and approximation			$E = F - A $ $= \left \int_0^{50} 62.5 \cdot 100 \cdot x \, dx - \sum_{i=1}^{10} 62.5x_i \cdot 100 \cdot \Delta x \right $	$\text{Error} = E = 7,812,500 - 8,593,750 = 781,250 \text{ pounds}$
Bound on the Error: The difference between an overestimate and underestimate, equal to the estimated force on the deepest strip.			$E < \text{overestimate} - \text{underestimate}$ $= \sum_{i=1}^{10} 62.5x_i \cdot 100 \cdot \Delta x - \sum_{i=0}^9 62.5x_i \cdot 100 \cdot \Delta x$ $= 62.5 \cdot 50 \cdot 100 \cdot \Delta x$	$E < 1,562,500 \text{ pounds}$
Method to Achieve Desired Accuracy: Use thin strips: estimated force on the deepest < allowable error			For an error bound of ϵ , choose $n > \frac{62.5 \cdot 50^2 \cdot 100}{\epsilon}$	For an error less than 2000 pounds, choose $n > 7812.5$

Results from these studies also indicate that the activities are effective in helping students systematize their reasoning around approximation ideas. Several of these students were able to make sense out of the epsilon-delta definition in terms of their approximation language at which point they began interchanging language and symbols related to approximation and the formal definition and referred to them as being the same thing. This is an indicator of structured, scientific reasoning since it is only possible if the student is able to recognize the underlying structure despite different sets of terminology.

In subsequent activities, students are given progressively fewer prompts for techniques such as finding under and overestimates and are expected to apply these techniques appropriately on their own. For each task, they are asked to identify contextual, graphical, numerical, and algebraic referents for each of the following questions:

- 1) What are you approximating?

- 2) What are the approximations?
- 3) What are the errors?
- 4) What are bounds on the size of errors? and
- 5) How can the error be made smaller than any predetermined bound?

Again, questions four and five are emphasized as reciprocal processes so that students see and remember the purpose of each. Throughout all of these materials, the structure of the underlying limit concepts determines the nature of the instructional activities. Further, answering these questions encourages students' explorations into the relevant structures of the concept defined in terms of limits. In activities about the derivative, the need to approximate a rate of change to a given quantity results in the exploration of average rates of change over small intervals and the analysis of underestimates and overestimates based on arguments of increasing or decreasing rate derived from the context. In the previous activity on the definite integral and the examples shown in the figure below, the structure of refinements to Riemann sums emerges as a result of engaging in the need to approximate to a given accuracy. Bob's quote at the beginning of this chapter in which he interpreted the definition of the derivative in terms of approximation is illustrative of the type of reasoning that has emerged consistently in the teaching experiments. Exploration, presentations, and discussion of multiple contexts exhibiting a common structure encourages the abstraction of the limit concept within the particular conceptual strand of calculus being covered. Figure 3 shows an example of such an activity for students to explore definite integral structures.

Summary

We have outlined several approaches to instruction related to limit concepts discussed in the mathematics education research literature. A typical class is often not represented by any one of these approaches but reflects a mixture of them. Regardless of the approach, however, the literature indicates students have major difficulties understanding limit concepts, which in turn impedes their understanding of other fundamental ideas in the calculus. In the first part of this chapter, we applied Piaget's theory of abstraction to characterize potential sources of student difficulties for various approaches. By highlighting these difficulties, we hope to assist individuals responsible for calculus instruction to address typical pitfalls. For example, an initial step in this direction might be to directly address common misinterpretations of imagery such as zooming in on a graph or viewing the fundamental theorem of calculus as being true as a result of an area collapsing in dimension to a line.

The second half of the chapter is intended to provide an example of designing an approach to calculus instruction that is coherent with respect to its treatment of limit concepts. The example provided uses common notions about approximations, is based on Piaget's theory of abstraction, and builds a structural understanding through repeated engagement in activities that reflect that structure. Certainly many other approaches could be designed to accomplish similar results. Our research shows that this design-based approach to instruction on limits develops a rich cognitive structure that reflects the standard mathematical definitions and applications and is powerful in supporting instruction on the other major concepts in calculus defined in terms of limits. This approach provides a facility with these major concepts grounded in ideas of

approximation and bounding error which are the basis for many applied applications of mathematics (e.g., in physics and engineering) and for a rich understanding of the mathematical formulas, theorems, and tools used in computational techniques. Finally, students are encouraged to develop an intuitive facility with the structures that can form a foundation for later abstraction to epsilon-delta and epsilon- N constructions, the basis of formalization and proof in upper-division and graduate analysis courses and of computational techniques in many applied mathematics and differential equations courses.

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