CONNECTING SCIENCE AND MATHEMATICS: THE NATURE OF PROOF AND DISPROOF IN SCIENCE AND MATHEMATICS

ABSTRACT. Disagreements exist among textbook authors, curriculum developers, and even among science and mathematics educators/researchers regarding the meanings and roles of several key nature-of-science (NOS) and nature-of-mathematics (NOM) terms such as proof, disproof, hypotheses, predictions, theories, laws, conjectures, axioms, theorems and postulates. To assess the extent to which these disagreements may exist among high school science and mathematics teachers, a 14-item survey of the meanings and roles of the above terms was constructed and administered to a sample of science and mathematics teachers. As expected, the science teachers performed better than the mathematics teachers on the NOS items (44.1% versus 24.7% respectively) and the mathematics teachers performed better than the science teachers on the NOM items (59.0% versus 26.1% respectively). Nevertheless, responses indicated considerable disagreement and/or lack of understanding among both groups of teachers concerning the meanings/roles of proof and disproof and several other key terms. Therefore it appears that these teachers are poorly equipped to help students gain understanding of these key terms. Classroom use of the *If/and/then/Therefore* pattern of argumentation, which is employed in this paper to explicate the hypothesis/conjecture testing process, might be a first step toward rectifying this situation.

KEY WORDS: nature of science, nature of mathematics, hypothesis, theory, axiomatic method, axiom, theorem, proof, disproof

INTRODUCTION

Consensus holds that science instruction should not only help students gain understanding of scientific concepts but should also help them better understand the “nature” of science. In other words, students should understand how science works, how scientists reason, and the epistemological status of scientific knowledge, that is the extent to which scientific claims can be “proved” or “disproved” (e.g., McComas & Olson, 1998). Unfortunately, scholarly disagreements remain regarding some NOS issues (e.g., Alters, 1997; Lawson, 2005; Smith, Lederman, Bell, McComas & Clough, 1997) some of which have made their way into curricular materials. For example, in a review of U.S. high school biology textbooks, McComas (2003) found considerable disagreement among textbook authors regarding the meanings of several key NOS terms such as theory, law, hypothesis, prediction. Thus, not surprisingly, research reviewed by Lederman (1992), Dass (2005) and Akerson, Morrison and McDuffie (2006) indicate that students have relatively poor NOS understanding and that helping them improve that understanding is difficult at best.

The situation may be no better in mathematics. Mathematical proof is central to the nature of mathematical reasoning. Further, the National Council of Teachers of Mathematics in the U.S. (NCTM, 2000) states that mathematical instruction should enable all students to: a) recognize reasoning and proof as fundamental aspects of mathematics, b) make investigative mathematical conjectures, c) develop and evaluate mathematical arguments and proofs, and d) select and use various types of reasoning and proof. However, as pointed by Lin, Yang and Chen (2004), both high school and university level students have difficulty in producing mathematical proofs and even in recognizing what proofs are. Although some promising pedagogical approaches have been reported (e.g., Christou, Mousoulides, Pittalis & Pitta-Pantazi, 2004), most students, and perhaps even some of their teachers, lack an understanding of the role played by and the importance of proof in mathematics.

But what exactly does it mean to prove a mathematical claim? Consider that on any given day, mathematics students who may be trying to understand the proof process may also be attending a science class in which they encounter the scientific "method." The scientific method is also often described as a process of idea generation and test. However, science students may be taught that scientists cannot "prove" that any particular scientific claim is correct. Instead, they may be told that in science one can only "disprove" a claim. In fact some science teachers may even go so far as to state that even disproof is impossible. What are these students to think upon hearing that proof is possible in mathematics but not in science? Do mathematicians know something about truth finding that scientists do not?

Given this apparent lack of consensus among teachers, textbook authors and curriculum developers, and even among science and mathematics educators/researchers the first order of business should be one of clarification. Therefore, the next sections of this paper will introduce relevant examples from science and from mathematics that will hopefully explicate the nature of the idea testing process in both disciplines and determine the extent to which scientific and mathematical claims can said to have been proven either true or false, i.e., either be "proved" or "disproved." To clarify the similarities and differences between the scientific and mathematical reasoning used to test knowledge claims, all of the examples will be explicated using the *If/and/then/Therefore* linguistic pattern introduced by Lawson (2004). This linguistic pattern should be particularly effective because it presumably also underlies "everyday" thinking, thus should be relatively easy for both teachers and their students to assimilate (e.g., *If* that shiny spot in the road ahead really is a puddle of water, *and* I keep driving toward it, *then* my car's tires should make a splash. *But* when I drove ahead, the shiny spot disappeared and my tires didn't make a splash. *Therefore* the shiny spot was not a puddle.).

Following this clarification, the paper will introduce a series of statements that were administered to a sample of secondary science and mathematics teachers. The statements were designed to assess the extent to the key NOS and NOM processes and terms were understood. Based on previous reports one might expect that the teachers will demonstrate considerable disagreement as to the meanings of several key terms. If this expectation is confirmed, steps should be taken, first for teachers and then for their students, to help them better understand the meanings of these key terms and their roles, and limitations, in the knowledge-generating and testing process.

HOW ARE SCIENTIFIC CLAIMS TESTED?

Scientific reasoning is often characterized in terms of hypothesis generation and test. Thus, careful examination of this process should not only help clarify the nature of hypotheses, predictions and so on, but may also shed light on the epistemological status of scientific conclusions, that is the extent to which scientific "proof" and "disproof" are possible. Let's start with a relatively simple example.

Suppose you walk into a room full of swinging pendulums and notice that some are swinging faster than others. This strikes you as puzzling so you ask: Why do pendulums swing at different speeds? You have some experience with playground swings so you quickly generate three alternative hypotheses. Perhaps the amount of weight hanging at the end is the cause. Perhaps it's the string's length. Or perhaps swing speed depends on how far the weight is pulled back before its release. With these alternatives in mind you move on to their test. To test the string-length hypothesis you generate the following deductive argument: *If*... changing string length (p) causes change in swing speed (q), i.e., $p \supset q$, *and...* we change string length (p), *then...* swing speed should change (q). Logicians refer to this form of argumentation (i.e., $p \supset q$, $p \therefore q$) as modus ponens. Use of modus ponens gives us a logical conclusion (swing speed should change). Scientists refer to this as an expected result – a prediction. So we now conduct the experiment and observe its result, which is that the swing speed does change (q). Because this observed result (q) matches our prediction (q), we have support for the hypothesis ($p \supset q$). Thus the logic of this supportive argument spelled out using the *If/and/then/Therefore* linguistic pattern looks like this: *If...* $p \supset q$ *and...* p *then...* q. *And...* q. *Therefore...* $p \supset q$.

This argument form is known as affirming the consequent. Interestingly logicians consider arguments of this form to be logically fallacious (Hempel, 1966; Tidman & Kahane, 2003). The argument would be logically sound only if one cause were possible, in which case a bi-conditional relationship would hold (i.e., $p \supset q$ and $q \supset p$). But in causal contexts this is not necessarily the case. Thus we are left with the conclusion that scientific arguments with supportive evidence must remain somewhat suspect. Said another way, because any particular effect may have several causes, which may at times lead to the same predictions, when one conducts the test and observes the predicted result, one cannot be certain that the tested hypothesis alone was responsible. Hence scientific "proof" (in the sense of certainty of correctness) is *not* possible. In other words, the logic of the situation tells us that we can never be certain that our conclusions are true. But is scientific disproof possible? Can we be sure that any particular claim is false?

To see how scientific "disproof" this might work, let's return to the pendulum example and test the weight hypothesis using the following deductive argument: *If...* changing the amount of weight on the string (p) causes change in swing speed (q), i.e., $p \supset q$. *and...* we change the amount of weight (p), *then...* swing speed should change (q). This argument also employs modus ponens ($p \supset q$, $p \therefore q$). We now conduct the test and discover that the swing speed does not change (not q). Because this result (not q) does not match our predicted result (q), we can conclude that the weight hypothesis ($p \supset q$) has been contradicted. The entire contradictory argument can be summarized using the *If/and/then/Therefore* pattern like this: *If...* $p \supset q$ *and...* p *then...* q. *But...* not q. *Therefore...* not $p \supset q$.

This argument form can be slightly altered to read as follows: *If...* the weight hypothesis is correct, *and...* we change the weight, *then...* the swing speed should change ($p \supset q$). *But...*the swing speed does not change (not q). *Therefore...* the weight does not matter ($\therefore$not $p \supset q$). Stated in this way, we can identify a logically valid form of argumentation known as modus tollens (i.e., $p \supset q$, not q, $\therefore$not $p \supset q$). Thus we can see why Popper (1965) claimed that science proceeds through falsification (disproof), but not through proof. Nevertheless, as Woodward and Goodstein (1996) point out, scientists are not necessarily trying to find out that they are wrong. We all realize that one does not win a Nobel Prize for rejected hypotheses. To win the prize you need to be fortunate enough to have tested a hypothesis that turns out to be supported, even if it cannot be proven. But is disproof really possible in science? Based on the previous example, it might seem so. However before we draw that conclusion we should consider a somewhat more complicated example.

Suppose you take a walk in a park and observe two nearly identical trees. Tree A has tall grass growing under it, while Tree B has nearly none. How might this puzzling observation be explained? Hypotheses include: Tree B provides too much shade for grass growth; Tree B drops grass-killing fruit; children trample the grass under Tree B; and so on. Let's test one of these hypotheses. Suppose Tree B's branches are cut off permitting more sunlight to reach the ground. The too-much shade hypothesis leads to the prediction that the grass should now grow. Suppose we conduct the test and after several weeks we observe no grass growth. Have we, therefore, disproved the too-much shade hypothesis? In this context, the logic of modus tollens would read like this: *If*...too much shade causes grass to grow poorly, *and*...some of Tree B's branches are cut off permitting more sunlight to reach the ground, *then*...the grass under Tree B should grow ($p \supset q$). *But*...after the branches are cut off, the grass under Tree B does not grow (not q). *Therefore*...the explanation must be false. It has been disproved ($\therefore$ not $p \supset q$).

Would you draw this conclusion? Hopefully you would not. Too much shade may still be the reason that grass did not grow under Tree B, but perhaps the grass failed to grow after the branches were cut off because: a) we did not wait long enough; b) it was now too cold for the grass to grow; c) the soil now lacked sufficient water; d) no grass seed remained under Tree B, and so on. In other words, because we cannot identify and control all of the independent variables that might influence the outcome, some doubt must remain regarding the truth or falsity of the tested explanation. Therefore, the correct conclusion to draw when a mismatch between predictions and results occurs is that the tested explanation has not been supported. But it still might be correct. Thus, the tested explanation has *not* been disproved.

To summarize, whenever we find a mismatch between a predicted and an observed result, it may mean that the explanation is wrong. But the mismatch may also be due to a faulty test (e.g., an uncontrolled experiment). And because we can never be certain that we have in fact conducted a perfect test, in spite of our best efforts, we can never be certain that the mismatch is due to the explanation and not the test. [1] Thus, like finding a scientific claim to be true (i.e., proof) is not possible, finding that a scientific claim is false (i.e., disproof), is also not possible.

THE ROLE OF PROOF AND DISPROOF IN MATHEMATICS

Based on the previous arguments we conclude that proof and disproof are not possible in science. However are proof and disproof possible in mathematics? To answer this question, we will employ the same *If/and/then/Therefore* linguistic pattern and begin by considering a familiar mathematical relationship.

Suppose you have a right triangle that is 3 units long on one of the sides adjacent to the right angle (call it side a) and 4 units long on the other adjacent side (call it side b). Now it turns out that the length of the third side (side c), called the hypotenuse is 5 units long. Further, it turns out that $3^2 + 4^2 = 5^2$ (i.e., 9 + 16 = 25). In other words, for this particular right triangle: $a^2 + b^2 = c^2$. This is an amazing discovery. But does the descriptive pattern hold for other right triangles, perhaps for *all* right triangles? For example, does it hold for a right triangle in which side $a = 4$ units and side $b = 6$ units? We can easily find out, i.e.: *If…* $a^2 + b^2 = c^2$ holds for all right triangles (descriptive hypothesis), *and…*$a = 4$ and $b = 6$, *then…*c should equal the square root of 16 + 36 = 52, or about 7.2 units. *And…*when one measures c it turns out to be very close to 7.2

[1] The mismatch of expectations and observations may also stem from a faulty deduction. For example, to explain the rise of water level inside an inverted cylinder placed over a burning candle sitting in a pan of water, students often think that the water rises because the flame burned up the cylinder's oxygen. To test this hypothesis they vary the number of burning candles and argue that: *If*... their hypothesis is correct, *and*...they vary the number of burning candles, *then*... the water should rise to the same level each time. Students make this deduction because they are assuming that more candles will not burn up more of the available oxygen. The additional candles will just burn it up faster. But this deduction does not necessarily follow as more candles may cause more air turbulence, hence lead to a greater mixing of the oxygen, hence lead to more burning, hence lead to more water rise.

units. *Therefore*…we have support for the descriptive hypothesis. Further, *if*… $a^2 + b^2 = c^2$ holds for all right triangles, *and*…$a = 6$ and $b = 9$, *then*…c should equal the square root of $36 + 81 = 117$, or about 10.8 units. *And*…when we measure c it turns out to be very close to 10.8 units. *Therefore*…we have additional support for the descriptive hypothesis.

Presumably if you measure some additional right triangles, you will find that the pattern holds for them as well. However, if you were to find just one right triangle in which the pattern did not hold, you could confidently conclude that the descriptive hypothesis (mathematicians would call it a conjecture) that $a^2 + b^2 = c^2$ holds for all right triangles is wrong. But suppose you measure lots of right triangles and find no exceptions. This might convince you that the descriptive hypothesis, the conjecture, is correct. But notice that you cannot observe *all* right triangles. How then can you find out if the pattern does in fact hold for all right triangles?

Mathematicians know that the answer to this question is contained in what they call the Pythagorean theorem (e.g., www.arcytech.org/java/pythagoras/history.html, also see Cullen, 1996) and in the geometrical contributions of Euclid (circa 365-275 BC). Euclid, known as the father of geometry, wrote *Elements*, perhaps the most successful textbook in history. Euclid's *Elements* was the first book to formally present what is now known as axiomatic method, which relied on deduction to "prove" descriptive claims including the one above. Euclid and other ancient Greeks believed that an important benefit of using the axiomatic method and deduction was that if one's initial claims (premises) were true, and if one's deductive reasoning was sound, then one's conclusions must also be true. Hence, if one could construct a deductive argument in which the descriptive knowledge claim that you want to test (the conjecture) "falls out" as a conclusion, one will have proven that it must be generally true.

To appreciate what the axiomatic method is and how it works let's once again apply the *If/and/then/Therefore* linguistic pattern to understand what Euclid did. In Book 1 of *Elements* (there are 13 Books in all) Euclid listed 23 statements (called *definitions*) including these four: 1) A point is that which has no part; 2) A straight line is a line which lies evenly with the points on itself; 3) A surface is that which has length and breadth only; and 4) When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right.

Euclid then presented 10 additional statements (called *axioms* or *postulates*) that he considered self-evident and intuitively obvious, thus in need of no further justification, such as: 1) Any two points can be joined by a straight line; 2) Any straight line segment can be extended indefinitely in a straight line; 3) Things that are equal to the same thing also equal each other; and 4) If equals are added to equals, then the wholes are equal.

Next Euclid advanced several additional knowledge claims that he only suspected were true. These were his conjectures. He then set out to use his previously stated definitions and axioms to construct what he hoped would be convincing step-by-step, well-reasoned, deductive arguments (i.e., these arguments are called *proofs*) that would conclude with the knowledge claim in question. Hence it will have been proven and will pass from conjecture to *theorem.*

In Book 1 Euclid used the axiomatic method to prove 48 conjectures, including these two: 1) A straight line standing on a straight line makes either two right angles or angles whose sum equals two right angles, and 2) When two angles of a triangle are equal, the sides opposite those angles are equal in length.[2] For example, Euclid's proof of the first conjecture consists of the following four steps and their respective deductive arguments:

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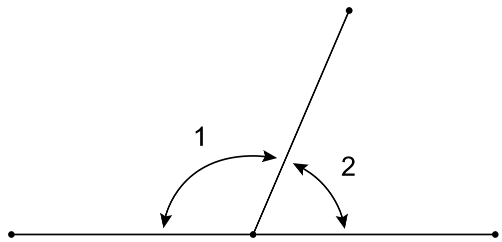
[2] One or more of Euclid's conjectures may strike some as intuitively obvious. Accordingly, Euclid could have listed them as axioms. However, one of Euclid's goals was to keep the number of axioms to a minimum and to prove as

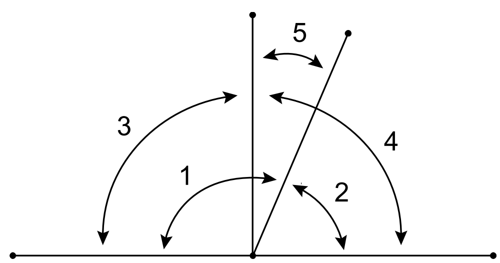
Step 1. Begin with a straight line standing on another straight line (the base line), which makes angles 1 and 2 (see below).



If angle 1 equals angle 2*,* then they are two right angles, i.e.:

> *If*…when a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is defined as a right angle (Definition 5), *and*…angle 1 and angle 2 are two adjacent and equal angles created by a straight line standing on another straight line, *then*…angles 1 and 2 are right angles.

Step 2. If angles 1 and 2 are not two right angles, then draw a third straight line at right angles to the base line (see below).[3]



Now angles 3 and 4 are both right angles. Note that angle 4 equals the sum of angles 2 + 5. Now add angle 3 to angles 2 + 5 and note that the sum of angles 2 + 3 + 5 equals the sum of angles 3 + 4 (i.e., 2 + 3 + 5 = 3 + 4), i.e.:

> *If*…when equals are added to equals, the wholes are equal (Axiom 7), *and*….angle 3 is added to angles 2 + 5 as well as to angle 4, which is equal to angles 2 + 5, *then*…the sum of angles 2 + 3 + 5 equals the sum of angles 3 + 4 (i.e., 2 + 3 + 5 = 3 + 4).

Step 3. Now add angle 2 to angles 3 + 5. Note that the sum of angles 2 + 3 + 5 equals the sum of angles 1 + 2 (i.e., 2 + 3 + 5 = 1 + 2), i.e.:

> *If*…when equals are added to equals, the wholes are equal (Axiom 7), *and*….angle 2 is added to angles 3 + 5 as well as to angle 1, which is equal to angles 3 + 5, *then*…the sum of angles 2 + 3 + 5 equals the sum of angles 3 + 4 (i.e., 2 + 3 + 5 = 3 + 4).

Step 4. Recall that 3 + 4 also equals 2 + 3 + 5. Thus, the sum of angles 3 + 4 equals the sum of angles 1 + 2 (i.e., 3 + 4 = 1 + 2). Also recall that angles 3 and 4 are both right angles. Thus, the sum of angles 1 + 2 also equals two right angles, i.e.:

> *If*…things that are equal to the same thing equal each other (Axiom 6), *and*….both 3 + 4 and 1 + 2 = 2 + 3 + 5, *then*…3 + 4 = 1 + 2.

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many additional conjectures from them as possible. Thus, as we shall see, because these conjectures could be proved from other axioms, when proven, they were more properly considered to be theorems.

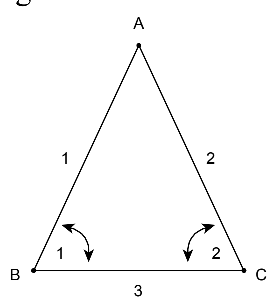
[3] To draw such a line, one follows procedures specified by a previously proved theorem.

Thus a straight line standing on a straight line makes either two right angles (i.e., angles 3 and 4) or angles whose sum equals two right angles (i.e., angles 1 and 2). In other words, Euclid's conjecture has been proven and it now constitutes a theorem. And as a theorem (i.e., a new "truth"), it can now be used in the construction of additional proofs to generate additional theorems (i.e., additional "truths").

The Role of Disproof in Mathematical Reasoning

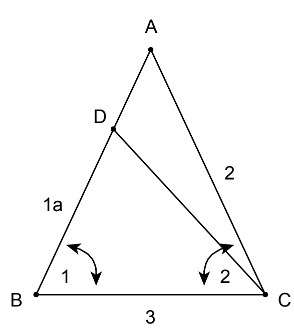
Next let's consider Euclid's proof of the conjecture that, "When two angles of a triangle are equal, the sides opposite those angles are equal in length." This conjecture was selected because it was the first of several conjectures that Euclid proved using a type of "backwards" reasoning in which he started not with the initial conjecture that he planned to prove, but instead with a counter conjecture that he planned to disprove. Basically, this reasoning pattern, which is sometimes called reasoning to a contradiction, proof by contradiction, or *reductio ad absurdum*, is designed to show that the counter conjecture leads deductively to contradictory (i.e., absurd) consequences. Therefore it must be false (i.e., has been *disproved*) and the initial conjecture must be true.

The conjecture proposes that when two angles of a triangle are equal, the sides opposite those angles are equal in length. For example, in triangle ABC below, angle 1equals angle 2, thus side 1 and side 2 are equal in length.



Euclid's argument[4] begins with the counter conjecture that side 1 is in fact longer than side 2 and proceeds as follows:

If…side 1 is longer than side 2 (counter conjecture), *and*...we mark off a segment of side 1 from point D to point B (call it segment 1a) equal in length to side 2, and we join points D to C (as shown below):

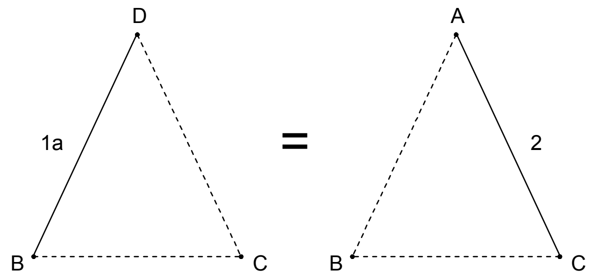


then…we should get a smaller triangle DBC and a larger triangle ABC with the

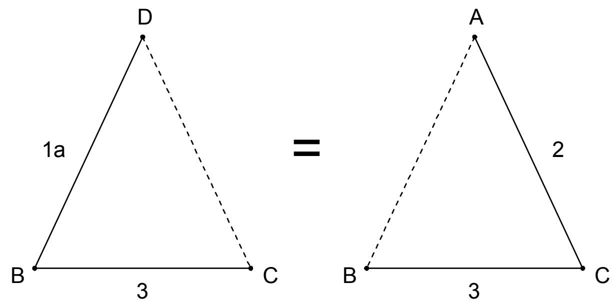
[4] In order to explicate the overall reasoning pattern, some of the intermediate deductive steps have been omitted.

following properties (some of which are contradictory):

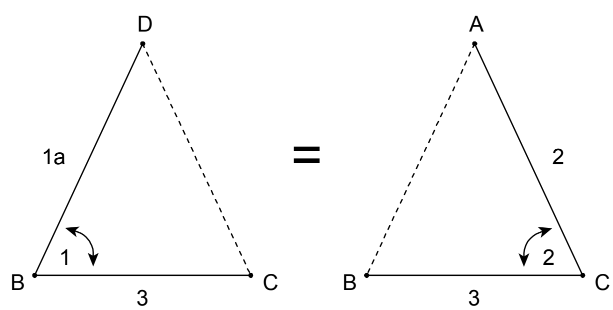
a) Side 1a of the smaller triangle should be equal in length to side 2 of the larger triangle (because that is the way they were drawn).



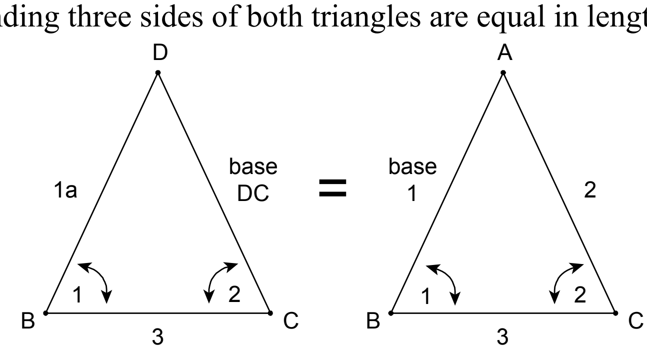
b) Side 3 should be in common to both triangles (because the two triangles overlap along that side).



c) Angle 1 of the smaller triangle should be equal to angle 2 of the larger triangle (because that is the way they were drawn).



d) Base DC of the smaller triangle should be equal in length to base 1 of the larger triangle (the bases should be equal given that the opposite angles are equal as are the lengths of the other two sides of both triangles). And the smaller triangle DBC should be equal in size to the larger triangle ABC (the sizes should be equal given that the corresponding three sides of both triangles are equal in length).



But… base DC of the smaller triangle cannot be equal in length to base 1 of the larger triangle (see the initial drawing of both triangles), and a smaller triangle cannot be equal in size to a larger triangle. In other words, these consequences are contradicted because they do not match what we previously know to be true. *Therefore*…the initial counter conjecture that side 1 is longer than side 2 must be false. Instead, side 1 and side 2 must be equal whenever their opposite side angles are equal (conclusion).

Can the Axiomatic Method Really Prove and Disprove?

Although there is little doubt that Euclid and mathematicians following in his footsteps believed that the axiomatic method yields truth, most modern day philosophers and mathematicians take a less certain view of the resulting knowledge (e.g., Kline, 1967; Lakatos, 1976; Hersh, 1997). Clearly if one begins with faulty assumptions (i.e., axioms/postulates) then the conclusions will also be faulty. And given that it would seem that one cannot be certain that one's assumptions are correct then certainty is lost.[5] Also can one be certain that each deductive step is correct? Although it may seem that deductive reasoning flows from some sense of logical "necessity" this is most likely an illusion. Instead, deductive logic seems to depend at least in part on one's knowledge of the context. To clarify this point, try to complete the following series of deductive arguments:

1) *If* stick A is longer than stick B *and* stick B is longer than stick C *then* stick A is…
2) *If* all men are mortal *and* Socrates is a man *then* Socrates is…
3) *If* the three angles of a triangle sum to 180 degrees *and* one of the angles is 90 degrees *then* the other two angles sum to a total of…
4) *If* this hose-like object is the trunk of an elephant *and* a blind man feels his way to the other end *then* he should feel…
5) *If* this hose-like object is the nose of a Glomp *and* a blind man feels his way to the other end *then* he should feel…

Most people have little difficulty deducing the correct conclusion in statements 1) through 4) presumably because they have the requisite declarative knowledge of sticks, men, triangles, and elephants. But when they come to statement 5), they are unable to generate a conclusion because they do not know about the nature of Glomps. The point is that deductive reasoning appears not to hinge on the use of general, all purpose infallible logic. Rather it more likely depends on the presence or absence of specific declarative knowledge. Consequently, just as in science, it would follow that use of the word "proof" and the claim of certainty in mathematics are not warranted. Interestingly, this realization took until the creation of non-Euclidian geometries in the 19th century to take hold. As Kline (1967) put it:

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[5] Indeed, during the 1930s, with his incompleteness theorems mathematician Kurt Gödel proved that you cannot be certain that your axioms are consistent. In other words, they might contradict each other in some very subtle way. You may eventually discover a contradiction, in which case the whole system is invalid - unless it can be revised to remove the contradiction, but you would still have to revisit everything that had previously been proven from the axioms. Or you may never discover a contradiction. Thus, there are two possibilities: a) there is a contradiction but you have not found it, b) there is no contradiction. Gödel's result shows that you cannot tell the difference between a and b, because you cannot prove that the system is self-consistent - even if it is (e.g., http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Godel.html).

> The belief that mathematics offers truths was firmly held by every thinking being until the creation of non-Euclidian geometry. But if several geometries which contradict one another all fit physical space, then it becomes very obvious, indeed, that all of these cannot be the truth, and, worse yet, one can no longer be sure that any of these is true. (p. 472)

In a similar vein, Albert Einstein summed up the prospects of ever achieving certainty this way: "Whoever undertakes to set himself up as a judge in the field of Truth and Knowledge is shipped wrecked by the laughter of the gods." (quoted in Kline, 1985, p. 207) And more recently, Hersh (1997) echoed the sentiment like this: "Mathematicians want to believe in unity, universality, certainty, and objectivity, as Americans want to believe in the Constitution and free enterprise, or other nations in their Gracious Queen or their Glorious Revolution. But while they believe, they know better (p. 39)."

We will now turn to the present study, which as mentioned, was aimed primarily at assessing the extent a sample of high school science and mathematics teachers understand several key NOS and NOM terms including proof and disproof (as just explicated).

METHOD

Subjects

Subjects were 45 high school (grades 9-12) teachers. The teachers (22 science teachers and 23 mathematics teachers, mean = 7.8 years of teaching experience, range 1 to 28 years) were from four school districts located in four suburban cities located in the southwest United States. They were taking part in a National Science Foundation supported Mathematics Science Partnership Program to improve secondary school science and mathematics teaching. The program consists of a sequence of four graduate level in-service courses that attempt to connect mathematics, science and engineering concepts and processes using an inquiry-based mode of instruction. Although the sample was not randomly drawn, we have no reason to believe that the teachers are atypical of U.S. science and mathematics teachers.

The Nature of Science and Nature of Mathematics Survey

The survey (see Table I below) consists of 14 items concerning the meanings and roles of proof, hypotheses, predictions, theories and laws in science and the meanings and roles of proof, conjectures, axioms, theorems, definitions, and postulates in mathematics. The survey was administered during about 20 minutes of class time at the beginning of the third course called *Connecting Biology, Geology and Mathematics.* Teachers were asked to read each item and respond on a five-point Likert scale indicating whether they strongly disagreed, disagreed, did not know, agreed, or strongly agreed with each item. Thus, survey responses are based on one's understandings of the terms used in each item.

Survey Validity. The survey does not directly assess one's ability to reason scientifically or mathematically. However, some of the survey items were previously administered to a sample of pre-service biology teachers enrolled in a teaching methods course as a pre- and post-test (Lawson, 2003). The intent of that study was to determine the extent to which the course influenced changes in student NOS understanding. In spite of the fact that students were not told which items were true and which were false, a comparison of pre- to post-test responses indicated considerable improvements on several items indicating improved NOS understanding.

As predicted, the extent to which improvements occurred was highly correlated with students' reasoning ability. Thus the survey items appear to be an indirect measure of reasoning ability based on the hypothesis that learning the meanings and roles of the terms as a consequence of instruction requires that one know how to reason scientifically. This hypothesis is consistent with the generally accepted relationship between language and thought. In other words, one's thoughts precede the linguistic expression of those thoughts in the sense that we have thoughts that can subsequently be expressed only approximately in words. Thus when thoughts (i.e., concepts or reasoning patterns) are expressed in either written or spoken language, as they are in the present survey items, the reader or listener must subconsciously generate and test hypotheses about the intended meanings (Gleitman & Papafragou, 2005). Consequently, when written language intends to convey meaning about one's reasoning patterns, a measure of one's reasoning ability should and did correlate highly with pre- to post-test gains. Hence, this result establishes predictive validity of those survey items.

Most likely one would gain additional insights into each teacher's NOS and NOM conceptions by use of open-ended questionnaires such as the *Views of Nature of Science* version B (Lederman, Abd-El-Khalick, Bell, & Swartz, 2002) or with semi-structured individual interviews. However, the present survey method has the advantage of being brief and easy to administer. Further, based on results of 1) published papers including the Lawson (2003) study just described, 2) discussions with practicing scientists and mathematicians, and 3) subsequent discussions with several of the participants of the present study, it appears that the survey items have face validity.

Scoring - Appendix A provides a scoring rationale for each item.[6] For each item, the correct response was either strongly disagree or strongly agree. When the correct response was strongly disagree, both it and the disagree response were scored correct. When the correct response was strongly agree, both it and the agree response were scored correct. Thus, primarily to simplify scoring and data analysis, present scoring scheme does not reflect the strength one agreement or disagreement with each survey item.

Insert TABLE I about here.

RESULTS

Table II lists percentages of responses in each category for each survey item for the science teachers. As expected, considerable differences of opinion are indicated. For example, 36% of the science teachers strongly disagreed or disagreed with Item 1 (A hypothesis is an educated guess of what will be observed under certain conditions) as opposed to 52% who agreed or strongly agreed. Similar wide differences of opinion were evidenced on virtually all of the science items with the exception of Item 6. Most science teachers (82%) agreed or strongly agreed with Item 6 (A hypothesis that gains support becomes a theory). This relatively high degree of consensus may appear to be positive until one realizes that this statement is incorrect (e.g., McComas, 2003). The science teachers' understanding of the mathematics items (Items 8 –14) was quite limited as evidenced by the high percentages of "Don't know" responses on each item (41% on Items 11 and 13 to 86% on Items 8 and 9). Clearly the science teachers have little understanding of the meaning of terms such as conjectures, axioms and postulates.

[6] Given that disagreements, or at least inconsistencies in usage, persist in the science and mathematics education literature regarding the meanings of some key NOS and NOM terms, a rationale is provided to justify the selected meanings.

However, some science teachers appear to have some understanding of the mathematical proof process and the meaning of the term theorem as 41% agreed or strongly agreed with Item 10 (Constructing an accepted mathematical proof yields a theorem) and 45% agreed or strongly agreed with Item 15 (A theorem is a consequence of logical argument built from axioms or other theorems).

Insert TABLE II about here.

Table III lists percentages of responses in each category for each survey item for the mathematics teachers. Generally, the mathematics teachers tended to agree or strongly agree with the science items. The only items that provoked wide disagreement were Item 3 (Hypotheses/theories cannot be proved to be true beyond any doubt) where 39% strongly disagreed or disagreed versus 48% who agreed or strongly agreed, and Item 4 (Hypotheses/theories can be disproved beyond any doubt) where 30% strongly disagreed or disagreed versus 60% who agreed or strongly agreed. More frequent differences of opinion surfaced on the mathematics items. For example, 39% strongly disagreed or disagreed with Item 8 (Conjectures, when proven, become axioms) versus 26% who agreed; and a surprising 35% said they did not know. Opinion was also split on Item 10 (Constructing an accepted mathematical proof yields a theorem) with 30% disagreeing versus 56% agreeing or strongly agreeing, on Item 11 (A postulate is a statement to be proven) with 57% strongly disagreeing or disagreeing versus 31% agreeing or strongly agreeing, and on Item 13 (A definition is a statement that has been proven) with 52% strongly disagreeing or disagreeing versus 39% agreeing or strongly agreeing. Item 14 (A theorem is a consequence of logical argument built from axioms or other theorems), was the only math item in which a clear consensus was reached (87% agreed or strongly agreed).

Insert TABLE III about here.

Figure 1 displays the percentages of correct responses on each item for the science teachers. Correct percentages on the science items varied from a low of 18% (on Item 6) to a high of 68% (on Item 5). Correct percentages on the mathematics items varied from a low of 0% (on Item 8) to a high of 45% (on Item 14). Collective performance on the science items (mean correct = 44.1%) was higher than collective performance on the mathematics items (mean correct = 26.1%).

Insert Figure 1 about here.

Figure 2 displays the percentages of correct responses on each item for the mathematics teachers. Correct percentages on the science items varied from a low of 9% (on Item 1) to a high of 48% (on Item 3). Correct percentages on the mathematics items varied from a low of 39% (on Item 8) to a high of 87% (on Item 14). Collective performance on the mathematics items (mean correct = 59.0%) was higher than collective performance on the science items (mean correct = 24.7%). Not surprisingly, the science teachers performed better than the mathematics teachers on the science items (44.1% versus 24.7% respectively) while the mathematics teachers performed better than the science teachers on the mathematics items (59.0% versus 26.1% respectively).

Insert Figure 2 about here.

DISCUSSION

The wide difference of opinion and relatively low levels of performance of both the science and the mathematics teachers on several of the survey items certainly does not bode well for these teachers' ability to effectively teach these key elements of NOS and NOM. One could argue that if teachers do not understand and agree on and the meanings and roles of key terms such as hypothesis, prediction, theory, postulate, axiom, and proof one cannot reasonably expect their students to gain related NOS and NOM understanding. The issue is compounded considerably when several key terms are defined and used differently across disciplines. As McComas (2003) put it: "Students are to be forgiven if they fail to see the distinction between the way that terms are used in common language in various disciplines but teachers should be more attentive to the potential for confusion." (p. 152)

Thus the next order of business should be to reach consensus among science and mathematics educators/researchers regarding the use of these key terms. However, if consensus cannot be reached, at least we should realize when terms are defined differently across disciplines and we should point out and try to clarify these differences. If progress in teaching NOS and NOM is to be made, the present differences should be resolved. To reiterate, the primary goal is to help science and mathematics teachers better understand their own discipline as well as each other's discipline so that they will not work at cross purposes by inadvertently defining key terms in different ways such that students not only fail to develop NOS and NOM understanding, but they also they fail to understand how science and mathematics connect.

CONCLUSIONS AND IMPLICATIONS

Neither the scientific method nor the axiomatic method yield certainty of their respective explanatory and descriptive conclusions. Proof is not possible in science because any particular effect may have several causes (i.e., multiple explanations), which may at times lead to the same predicted result. Hence, when one conducts a test and observes the result predicted by the hypothesis, one cannot be certain that the hypothesis alone was responsible. On the flip side, when predicted and observed results do not match, the fault may not lie with the tested explanation. Instead it may lie with a faulty test or with a faculty deduction. And because one can never be certain that one or the other of these faults has not occurred explanations cannot be rejected with certainty. The situation is similarly compromised in mathematics as both mathematical proof and disproof rely on the soundness of one's initial premises and one's deductive reasoning, which we have seen are both open to error.

On the other hand, in spite of this inherent uncertainty, we should not leave students with the idea that application of the scientific and axiomatic methods do not result in useful knowledge. Indeed the histories of science and mathematics, as well as the technological advances that can at least in part be attributed to the application of scientific and mathematical knowledge constitutes convincing evidence to the contrary. Nevertheless, it would seem that science and mathematics teachers, curriculum developers and textbook authors owe it to students to more carefully explicate the similarities, differences, and limitations of knowledge generation processes in both fields, particularly the meanings of the terms proof, disproof, hypotheses, predictions, theories, laws, conjectures, axioms, theorems, and postulates so that students have a better chance of avoiding misconceptions and/or confusion about how these aspects of science and mathematics work. The results reported in this paper suggest that at this time many science and mathematics teachers have too little understanding of the knowledge generation process in their own discipline, much less adequate understanding of the process in their sister discipline to help students acquire meaningful NOS and NOM understanding. If the present sample is representative of the general population of science and mathematics teachers it will be important for researchers and curriculum developers to collectively to tackle these issues so that meaningful teacher and student progress can be made.

A good place to start might be with the examples provided in this paper because, unlike most examples, the present examples all employ the same *If/and/then/Therefore* linguistic pattern that presumably lies not only at the heart of both scientific and mathematical reasoning, but also at the heart of “everyday” reasoning. Both developmental theory and experience suggest that an effective teaching approach is to 1) provide students with several opportunities to test knowledge claims, 2) ask them to reflect on their procedures and their reasoning patterns, 3) identify successful procedures and reasoning patterns, 4) introduce the *If/and/then/Therefore* linguistic pattern in the context of several of their arguments, 4) introduce the relevant scientific and/or mathematical terminology (e.g., hypotheses, predictions, conjectures, theorems). By *not* introducing the terminology first, which is more common, the students view their goal as one of testing claims and not one of memorizing new terms. Also once students internalize the relevant processes, the terms can be successfully introduced because the students have initial ideas to connect the terms with (i.e., ideas first – terms second). And lastly, 5) experience also suggests that challenging students to further reflect on their reasoning in new contexts and to construct written arguments utilizing the *If/and/then/Therefore* pattern, although difficult, is an effective means of helping them develop their abilities to reason scientifically and mathematically.

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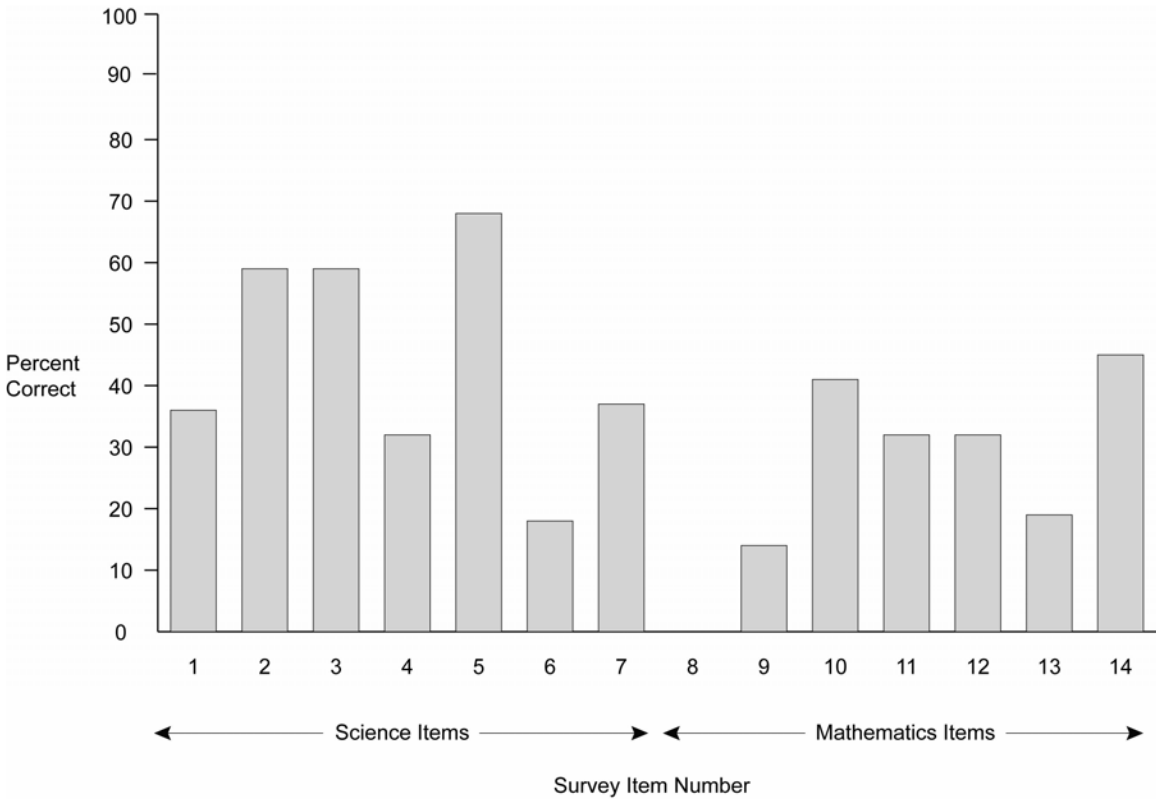

Figure 1. Percent of survey items correct for the science teachers (n = 22).

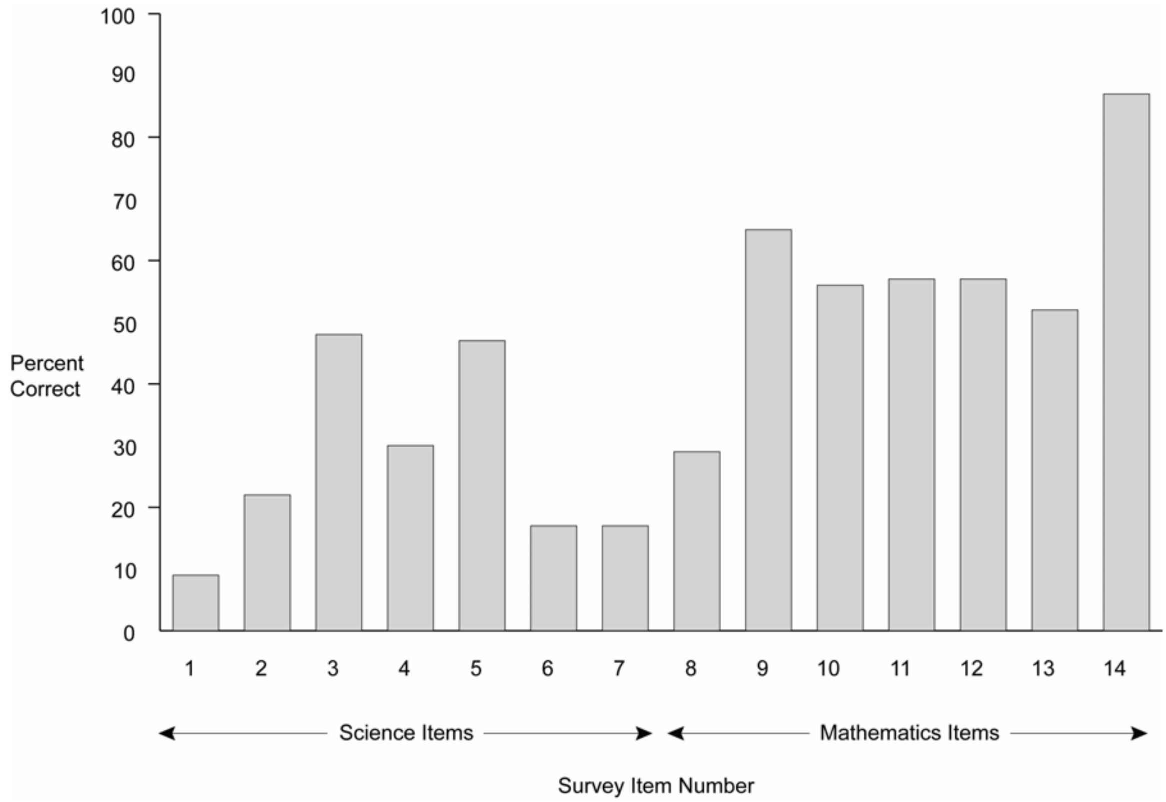

Figure 2. Percent of survey items correct for the mathematics teachers (n = 23).

TABLE I

The Nature of Science and Nature of Mathematics Survey

Next to each item write the number that best reflects your current belief:

1=strongly disagree 2=disagree 3=don't know 4=agree 5=strongly agree

Science Items
_______1. A hypothesis is an educated guess of what will be observed under certain conditions.
_______2. A conclusion is a statement of what was observed in an experiment.
_______ 3. Hypotheses/theories cannot be proved to be true beyond any doubt.
_______ 4. Hypotheses/theories can be disproved beyond any doubt.
_______ 5. To test a hypothesis, you need a prediction.
_______ 6. A hypothesis that gains support becomes a theory.
_______ 7. A theory that gains support becomes a law.

Mathematics Items
_______ 8. Conjectures, when proven, become axioms.
_______ 9. Conjectures, when proven, become theorems.
_______10. Constructing an accepted mathematical proof yields a theorem.
_______11. A postulate is a statement to be proven.
_______12. Postulates and axioms are mathematical assumptions.
_______13. A definition is a statement that has been proven.
_______14. A theorem is a consequence of logical argument built from axioms or other theorems.

TABLE II

Responses of the Science Teachers (n=22) to the Survey Items

Item #					
			Response (%)		
Science Items	Strongly disagree	Disagree	Don't know	Agree	Strongly agree
1	9	27	9	27	27
2	14	45	5	27	9
3	14	27	0	32	27
4	14	18	5	32	32
5	9	23	0	45	23
6	9	9	0	68	14
7	14	23	0	50	14
Math Items					
8	0	0	86	9	5
9	0	0	86	14	0
10	0	5	55	36	5
11	14	18	41	18	9
12	0	0	68	18	14
13	5	14	41	27	14
14	0	0	55	36	9

TABLE III

Responses of the Mathematics Teachers (n=23) to the Survey Items

Item #					
			Response (%)		
Science Items	Strongly disagree	Disagree	Don't know	Agree	Strongly agree
1	0	9	4	57	30
2	0	22	4	61	18
3	4	35	13	26	22
4	13	17	9	43	17
5	0	17	35	43	4
6	4	13	13	65	4
7	4	13	26	57	0
Math Items					
8	22	17	35	26	0
9	4	13	17	48	17
10	0	30	13	39	17
11	35	22	13	22	9
12	4	17	22	22	35
13	17	35	9	35	4
14	0	0	13	52	35

APPENDIX A: SURVEY ITEM SCORING RATIONALE

Science Items

1. A hypothesis is an educated guess of what will be observed under certain conditions.
A hypothesis is defined as a tentative explanation for some puzzling phenomenon, i.e., a proposed cause. One can certainly observe the puzzling phenomenon, but typically one does not observe its cause. For example, water rises when a cylinder is inverted over a burning candle sitting in a pan of water. This phenomenon is puzzling and can be observed. However, one cannot observe the cause of the water rise, which presumably is due to molecules of hot air escaping from the cylinder and the external air's relatively greater density and pressure pushing on the external water's surface. Granted in addition to causal hypotheses, descriptive hypotheses also exist (e.g., All crows are black). However, even here one cannot observe all crows. Again what one can observe is its predicted consequence: If... all crows are black, and... we find one more crow, then... it should appear black. Preferred response = 1.

2. A conclusion is a statement of what was observed in an experiment.
A scientific conclusion is a statement regarding the relative support or lack of support for a tested hypothesis or theory. For example, suppose one advances the hypothesis that water rises in the inverted cylinder mentioned above because CO_2 is created by combustion and this newly created CO_2 dissolves more rapidly in water than the original O_2. To test this hypothesis one could compare the amount of water rise in two containers. One container would contain CO_2 saturated water while the other would contain normal water. The hypothesis leads to the prediction that the water should rise higher in the container with normal water because the excess CO_2 would dissolve in this water but would be "blocked" by the CO_2 saturated water in the other container. When the experiment is conducted, we find that the water rises to the same level in both containers. This result does not support the initial hypothesis. Therefore, the conclusion would be that the CO_2 dissolving hypothesis was not supported. In other words, one observes puzzling phenomena and one observes experimental results, but one does not observe hypotheses and conclusions. Preferred response = 1.

3. Hypotheses/theories cannot be proved to be true beyond any doubt.
Because any two hypotheses or theoretical claims may lead to the same predicted result, eventual observation of that predicted result cannot reveal which hypothesis or theoretical claim is correct. For this reason, supportive evidence cannot prove a hypothesis or theory correct. Preferred response = 5.

4. Hypotheses/theories can be disproved beyond any doubt.
Contradictory evidence can arise due to an incorrect hypothesis/theory, to a faulty test (e.g., one in which all other variables were not held constant), or to a faulty deduction. Because it is not possible to be certain that all other variables were in fact held constant, or that one's deductions are correct, contradictory evidence cannot disprove a hypothesis or theory. Preferred response = 1.

5. To test a hypothesis, you need a prediction.
Hypothesis testing requires the generation of a prediction and a comparison of the predicted result with the observed result. The prediction may be of the classic sort generated in controlled experimentation. For example: If...water rises in the inverted cylinder because oxygen has been consumed (hypothesis), and...water rise with one, two, and three candles is measured while holding all other variables constant (controlled planned experiment), then...the height of water rise should be the same regardless of the number of burning candles (prediction). The prediction may involve circumstantial evidence. For example: If…O. J. Simpson killed Nichol Brown Simpson (hypothesis), and…a sample of the blood found in O.J.'s Ford Bronco is compared with a sample of Nichol's blood (planned test), then…the two blood samples should match (prediction). Or the prediction may involve correlational evidence. For example: If…breast implants cause connective tissue disease (hypothesis), and…the incidence of connective tissue disease in a sample of women with implants is compared to the disease incidence in a matched sample of women without implant (planned test), then…the disease incidence should be higher in the implant group than in the non-implant group (prediction). As alluded to above (see Item 1), descriptive hypotheses also require predictions for their test. For example, suppose one generates the descriptive hypothesis that all swans are white. Testing this hypothesis requires the following reasoning and resulting prediction: If…all swans are white, and…I observe several

additional swans (planned test), then…they should all be white (prediction). Thus, regardless of the type of hypothesis being tested and type of evidence collected, hypothesis testing requires the generation of one or more predictions. Preferred response = 5.

6. A hypothesis that gains support becomes a theory.
Like hypotheses, theories are explanations of nature. Hypotheses attempt to explain a specific observation, or a group of closely related observations. Theories attempt to explain broad classes of related observations, hence tend to be more general, more complex, and more abstract than hypotheses. Consequently, a hypothesis, regardless of the amount of support that may be obtained, does not become a theory. Preferred response = 1.

7. A theory that gains support becomes a law.
Tested and accepted generalizations (i.e., laws) describe nature in terms of identifiable patterns (e.g., F = ma, more candles make more water rise, the sun rises in the east and sets in the west). Explanations (both hypotheses and theories) attempt to provide causes for such patterns. Regardless of the amount of support that an explanation may obtain, that explanation does not become description. Hence, theories do not become laws. Preferred response = 1.

Mathematics Items

8. Conjectures, when proven, become axioms.
The proof process in mathematics is one in which a tentative statement, called a conjecture (sometimes called a hypothesis), is first advanced. Next one attempts to construct a step-by-step, logically-sound, deductive argument based on the use of prior definitions, prior assumptions (i.e., axioms/postulates), or already proven statements (i.e., theorems) in which the initial conjecture fall's out as the argument's ultimate conclusion. If such a deductive argument can be constructed, the conjecture is said to have been proven, and is henceforth known as a theorem. Preferred response = 1.

9. Conjectures, when proven, become theorems.
Based on the above description of the proof process, this is a true statement. Preferred response = 5.

10. Constructing an accepted mathematical proof yields a theorem.
Based on the above description of the proof process, this is a true statement. Preferred response = 5.

11. A postulate is a statement to be proven.
As mentioned, postulates are statements that are assumed to be true. Thus they are not statements that one attempts to prove. Preferred response = 1.

12. Postulates and axioms are mathematical assumptions.
This is a correct statement. Preferred response = 5.

13. A definition is a statement that has been proven.
Definitions are not proven. Rather they constitute a starting point in the proof process (the so-called axiomatic method). For example, in Book 1 of *Elements*, Euclid began by listing 23 definitions including: 1) A point is that which has no part; and 2) A straight line is a line which lies evenly with the points on itself. Preferred response = 1.

14. A theorem is a consequence of logical argument built from axioms or other theorems.
This is a correct statement. Preferred response = 5.